THE SPIRIT

OF

MATHEMATICAL ANALYSIS,

AND ITS

RELATION TO A LOGICAL SYSTEM.

BY

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Of the merits of a work of which truth is the object, one cannot have an adequate idea or a perfect relish, without some acquaintance with the errors against which it is levelled, and which it is calculated to displace. With respect to others, the apparent merit of such a work will be apt to be in inverse proportion to the real. The better it answers its purpose, of making an abstruse subject plain, the more apt it will be to appear to have nothing in it that is extraordinary.

An observation that seems to contain nothing more than what every one knew already, shall turn volumes of specious and formidable sophistry into waste paper. The same book may succeed ill with different sets of people for opposite reasons; by the ignorant, who have no opinions about the matter, it may be thought lightly of, as containing nothing that is extraordinary; by the false learned, who have prejudices they cannot bear to have questioned, it may be condemned as paradoxical, for not squaring with these prejudices.

JEREMY BENTHAM. *Life by Dr Bowring*, p. 85.

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NOTICE BY THE TRANSLATOR.

It is the intention of the Translator of the following Essay, should he meet with due encouragement, to publish a series of translations from the mathematical works of *Prof. Ohm*, of which the present forms the commencement. These will embrace the whole of Mathematics from the earliest to the most advanced stage, including what is commonly termed *pure* Mathematics (Analysis), and their applications to Geometry and Mechanics. They have been written with an especial view to didactical purposes, for which they have been long employed by the Author, but they will also be found exceedingly well adapted for self-instruction when a proper teacher cannot be obtained. The present Essay however forms, as it were, the preface to the other works of *Prof. Ohm*, and is particularly addressed to those, who have already acquired considerable mathematical knowledge, or have read those works, in order to point out the fundamental idea which pervades all his mathematical writings.

The great clearness and precision manifested in these writings, their extreme simplicity and logical accuracy, made a forcible impression on the mind of the Translator while pursuing his own mathematical studies a few years ago, and he could not help contrasting these Treatises with the many vague, half-elaborated works in his own language. Few persons have indeed pursued the study of Mathematical Analysis with the same anxiety and power to improve the foundations upon which it rests, as *Prof. Ohm*. A life continually spent in instructing others has enabled him to test and retest his views by the best of touchstones,—the mind of the learner. It is now twenty-seven years since his labours of composition were commenced. During this time he has employed his works in his professorial lectures delivered to willing* and delighted pupils at various institutions, but for the last twenty-two years at the University at Berlin, and he is at present engaged and nearly overwhelmed by having to give instruction in Mathematics at the University, Military School, and Engineers' School at Berlin. He has lived to see his views, which were at first ridiculed, taught by himself at the chief mathe-

* The study of mathematics is as voluntary at Prussian Universities as that of Classics at Cambridge.

matical schools in Prussia, and by those who have been under his tuition, in various parts of that country, to an extent shewn by the rapid sale of his elementary works. And he has been encouraged and rewarded throughout by the rapid and steady progress of his pupils, and the enthusiasm with which he has seen them inspired for Mathematical Analysis,—a result mainly owing to the simple, rigorous, and philosophical, yet inviting dress with which he has clothed his subject.

Convinced of the advantages attendant upon a study of these works, the Translator was from the first anxious to make them accessible to his countrymen, to the greater part of whom, and especially that part to which they would be most valuable, they were in their original language, a sealed book. Circumstances which he had not foreseen have allowed him leisure to labour at a translation of these works, and he has applied himself to the task as to a labour of love, looking for his only reward to the good effects he so sanguinely anticipates. The works, to which an asterisk is prefixed in the list given at the end of this Essay, will all be speedily translated, although they will necessarily only appear at intervals, the length of which must of course depend on the reception which these translations may experience. The "System" (with the exception of the first volume,) will appear last of all, as considerable alterations are contemplated by the Author in several of the volumes already published, and the five last volumes are still unfinished.

The Translator has to apologize for coining a few new words, which were rendered necessary by the systematic discrimination of ideas in the present work. He hopes however that they will not be found objectionable, and that since they are explained as they arise, they will occasion no difficulties to the reader. Thus the "numeric equations" (*Zahlen-Gleichungen*) mentioned in the preface, p. vii., must be distinguished from those equations commonly termed "numeric*al*" (*numerische Gleichungen*) as having coefficients which are either positive or negative whole or fractional, or zero. The five forms $\pm\mu$, $\pm\frac{\mu}{\nu}$, and 0, $\{$where μ and ν are whole (absolute) numbers, and μ is not a multiple of $\nu\}$ have been classed under the generic name of "*actual* numbers", *quasi numeri in actu, non in re*, corresponding very nearly to "real" or "possible quantities" in the old nomenclature; while the designation "real number" has been confined to the positive whole number. The word *cypher* has also been employed more in accordance with its etymological signification (Hebrew, "çepher", mark, number,) to denote one of the characters 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0 (zero) and not, as it is frequently misapplied, confined to 0 (zero) only. "Value in cyphers" is nearly tantamount to what is commonly called "arithmetical value", which latter term is too general according to the present system, as implying *any* expression or form of number. A "denominate" number is one referred to a certain "denomination" or unit, and therefore represents a magnitude (see Appendix). On the other hand an "indenominate" number is one *not* referred to any particular unit, and is therefore the *abstract* number. Such verbs as *potentiate, radicate, logarithmate*, with their corresponding substantives will, it is hoped, be found so convenient as to meet with indulgence.

It will be perceived by the Author's Preface, that there is to be a *second* Essay, bearing the same title as the present; the first Essay is however complete in itself, comprising the whole of *general* (or formal) analysis. On account of the numerous and engrossing occupations of the Author, it may be some years before the second Essay is published, but as soon as it appears the Translator hopes to be enabled to lay it before the English public.

A. J. ELLIS.

COTMANDENE LODGE, DORKING,
April 20, 1843.

THE AUTHOR'S PREFACE.

There are certain wants, which, although we may more or less delay to satisfy them, we can never entirely repel. Among these is the want of philosophising, and we have thus an explanation of the fact that a new philosophical system is continually driving its predecessor into the province of the history of human intellect, while it has in turn to cede its own place to another that follows hard upon it. And thus the mathematician who is solely employed in the collection of new materials, generally looks down without the least sympathy, nay even sometimes with a proud contempt upon every endeavour to place mathematics upon a surer and more satisfactory foundation, while his own writings are frequently quite sufficient to shew that he himself has not been entirely able to repress the feeling that such a foundation is wanted.

The Author communicates in the following pages, as briefly as he has found it possible, the *nature* of those views, which he has taught in his several works since 1816, and especially since 1822, and which he still continues to teach,—views, which have had the good fortune to meet with considerable approval, as detailed in his various Instruction-Books, but which have also been very much misunderstood, and probably the more easily admitted of, being misunderstood, that an Instruction-Book has necessarily to take into consideration a great number of points which stand in the way of forming a comprehensive conception of the *nature* of the subject. The present little work presupposes that the reader will himself supply all that is merely practical, and is solely employed in exhibiting logically determinate, clear and well-defined ideas, all such ideas, namely, as form the pivots upon which mathematical analysis revolves. The reader is therefore requested to bestow some attention upon this representation, and to examine with some degree of care, whether he finds in it, as a whole, *that* which is commonly termed *internal connection and scientific unity*, or whether this representation does not at least approach the *idéal* which the reader may have formed of such consequentiality, nearer than any other view, which has as yet been exhibited, of mathematical analysis considered as a science.

In the present, first Essay, the Author has established the foundation of all calculation. He was obliged to designate the contents of mathematical analysis, as "the knowledge of the oppositions and relations in which the seven operations (i. e. sums, differences, products, quotients, powers, roots and logarithms,) stand to one another." These oppositions and relations are enunciated in *general equations,* between *general expressions,* in which the signs of operation constitute the *essence,* while the letters are only *supporters* of these signs of operation, so that these letters represent *neither magnitudes, nor numbers,* but are considered as *perfectly insignificant* (*inhaltlos*). These general expressions may be even infinite series, provided that they proceed according to the powers of any such *supporter,* i. e. provided that they have *the form of whole functions.*

From these *general* equations, on which the whole of mathematical analysis turns, and which may be also aptly termed *formal equations,* we proceed to distinguish those in which the letters have already received a signification, i. e. represent either, so-called, indenominate whole numbers, or at any rate such general expressions as originally owe their existence to whole numbers. These expressions, which originally owe their existence to whole numbers, may be reduced either to one of the 5 forms $\pm\mu$, $\pm\frac{\mu}{\nu}$, and 0, and are then termed *actual numbers,* or at any rate to the form $p + q \cdot \sqrt{-1}$ where p and q are such actual numbers, and are then termed *imaginary numbers,* or *imaginary expressions.* If we term such an equation in which the several letters are no longer mere supporters of operations, but have already such a signification as has just been pointed out, i. e. represent actual or imaginary numbers,—no longer a "general," but a *numeric equation,* it *follows* from the idea of a *general* equation, such as the Author has established, that the expressions on each side of the sign of equality (in such a numeric equation) represent either *one and the same* actual number a, or *one and the same* imaginary number $p + q\sqrt{-1}$, while the idea of a *general* equation must differ from the usual one; in order to deduce from it that the actual number a, or the imaginary number $p + q\sqrt{-1}$ is really represented by one of these expressions (to the left or right of the sign of equality). If infinite series occur in such a *numeric equation* they must be considered as convergent, while in the *general* equation infinite series will occur which cannot be termed either convergent or divergent, precisely because every letter is, in a *general* equation, only a supporter of the operations, and still perfectly insignificant.

This may as well be illustrated here by an example. In (sect. 68) of this Essay, we have proved that the binomial theorem holds perfectly generally, i. e. that the equation

$$(1)\quad (1+z)^n = 1 + n \, . \, z + \frac{n^{2\mathrm{I}-1}}{2\,!} \, . \, z^2 + \frac{n^{3\mathrm{I}-1}}{3\,!} \, z^3 + \frac{n^{4\mathrm{I}-1}}{4\,!} \, z^4 + in\ inf.^*$$

holds with perfect generality for every z and for every n, so that n and z may be considered as perfectly insignificant, and representing neither a magnitude nor a number, provided that we take for the expression on the left $(1+z)^n$, when n is not a positive whole number, only one single one of the expressions represented by this power, and that the correct one. If we put $\frac{z}{n}$ for n in this equation, it becomes

$$\left(1+\frac{z}{n}\right)^n = 1 + z + \frac{z^2}{2\,!} \, . \, \frac{n^{2\mathrm{I}-1}}{n^2} + \frac{z^3}{3\,!} \, . \, \frac{n^{3\mathrm{I}-1}}{n^3} + in\ inf.$$

so that the r^{th} term (after the very first) $= \frac{z^r}{r\,!} \, . \, \frac{n^{r\mathrm{I}-1}}{n^r}$. Now $\frac{z^r}{r\,!}$ is the r^{th} term (after the very first) of the series for e^z, if e represents the base of the natural logarithms; hence this development of $\left(1+\frac{z}{n}\right)^n$ would become the series for e^z if we could take n so that $\frac{n^{r\mathrm{I}-1}}{n^r}$, that is, $\frac{n\,(n-1)\,(n-2)\,(n-3)\ldots[n-(r-1)]}{n\,.\,n\,.\,n\,.\,n\ldots n}$,

$$\text{that is, } 1 \, . \left(1-\frac{1}{n}\right) . \left(1-\frac{2}{n}\right) . \left(1-\frac{3}{n}\right) \ldots \left(1-\frac{r-1}{n}\right)$$

should become equal to 1 for *every* positive whole number r. It is usually said that this is the case for $n = \pm\infty$, and it is therefore usually asserted that

$$\left(1+\frac{z}{n}\right)^n = e^z \text{ for } n = \pm\infty,$$

while no consideration has been perhaps paid to the fact, that this would be the case for *those* terms of the series for which r is not itself infinitely great, but that, so to speak, *in infinito*, the factor $\frac{n^{r\mathrm{I}-1}}{n^r}$ will always be indeterminate, although included

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* The symbol $n^{4\mathrm{I}-1}$ represents the product of 4 factors, of which the first is n, and each succeeding one formed from the preceding by the addition of -1, i. e. the product $n\,(n-1)\,n\,(n-2)\,(n-3)$; it is termed a *factorial*. In the same way the symbol $4\,!$ denotes the product $1\,.\,2\,.\,3\,.\,4$, and is termed a *faculty*.

between certain determinate limits. The above conclusion is therefore not perfectly correct, and the equation

$$\left(1+\frac{z}{n}\right)^n = e^z \text{ for } n = \pm\infty$$

cannot be admitted as a general (formal) equation. But if we consider z as no longer insignificant, but as either actual or imaginary, and therefore of the form

$$\rho \,.\, \cos\phi + \sqrt{-1} \,.\, \rho \,.\, \sin\phi,$$

$$\text{then } z^r = \rho^r \,.\, \cos r\phi + \sqrt{-1} \,.\, \rho^r \,.\, \sin r\phi,$$

and then $\frac{z^r}{r!}$ (both in the series found for $\left(1+\frac{z}{n}\right)^n$ when $n = \pm\infty$, and in the series for e^z) would approach zero the more nearly, the greater r were taken, i.e. the further the terms were taken on towards infinity. And hence at the moment that the assertion that

$$\frac{n^{r|-1}}{n^r} = 1 \text{ for } n = \pm\infty$$

ceases to be correct, the terms which are influenced by this incorrect assertion, may be considered as *equal to zero*, and hence these two discrepancies correct one another. Hence although the equation

$$(2) \quad \left(1+\frac{z}{n}\right)^n = e^z \text{ for } n = \pm\infty$$

is not correct as a general (formal) equation, it is yet not incorrect for any actual or imaginary value of z as a numeric equation, and that, because both series, that for $\left(1+\frac{z}{n}\right)^n$, and also that for e^z are *convergent* for any such value of z.* Finally if the terms of these series had not these large denominators $r!$ (that is, $1.2.3.4.5.6...r$) which cause the series to converge for *any* value of z, the above-mentioned equation would have only held for those values of z for which the said series would have been convergent.

Hence while the equation (No. 1) holds with perfect generality, when z is considered as perfectly insignificant, being a mere supporter of the operations, and when therefore we cannot speak of the convergence or divergence of the series, we perceive

* Hence if we wish to calculate the value of e^z approximately for any positive or negative values of z, we have only to calculate the value of $\left(1+\frac{z}{n}\right)^n$ for any (disregarding the sign) very large positive or negative, whole or broken, rational or irrational n, and the approximation will be the closer the greater n is taken. For example, we can in this manner calculate the number e itself.

in (No. 2) an equation which is only correct as a numeric equation, i. e. which is only correct when the series are convergent.

A *definite Integral* always presupposes numeric values; consequently equations in which definite integrals occur are seldom or never correct as general (formal) equations, but can only be admitted as numeric equations; consequently the convergence of any infinite series which may occur in them is an indispensable condition, whereas the condition of convergence with respect to a general series in general investigations, such as must be necessarily first established as the foundation of the possibility of any calculation, is quite as absurd, as if, upon an investigation of the capabilities of a living and yet powerful man, the condition were premised as indispensable,—that he should be already dead.

But if equations can occur which no longer hold generally but only upon the particular hypothesis that they are numeric equations, the theory of such numeric equations must be established for itself in a second Essay, and the "Theory of Definite Integrals" (in which numeric equations first occur in considerable numbers,) will have determinately and distinctly to solve the problem of discovering a method of calculating with such equations. The Author believes that he has now shewn the Reader in the clearest possible manner, what he will have to expect from a second Essay. While namely this first Essay treats of perfectly general forms, as the first and most necessary foundation of all calculation, the second Essay will have to continue these general investigations, but at the same time to bestow more attention upon the passing of general into particular and numeric forms, which passing takes place when the former are considered upon determinate and particular hypotheses.

Among the *practical results*, which have been the consequences of the view here established, and which are to be found at once in the present first Essay, the Author thinks that he ought to distinguish the following:

(*a*) A *fully assured* method of calculating with roots in general, and with imaginary expressions in particular;

(*b*) The establishment of those formulæ, which must take the place of the usual rules

$$a^x . a^z = a^{x+z}; \quad a^x : a^z = a^{x-z}; \quad (a^x)^z = a^{xz}; \text{ and so on,}$$

in order that we may calculate with general powers and logarithms *in perfect safety*, inasmuch as the above formulæ which are those usually employed, are only partially correct;

(c) A *fully assured* method of calculating with such infinite series as are yet perfectly general, and precisely for that reason *not* convergent.

In the "Tracts upon some parts of higher Mathematics" (*Aufsätze aus dem Gebiete der höheren Mathematik, Berlin*, 1823), will be found some applications of these elementary, but assured methods of calculation, especially in the last of those Tracts, to which the Author thus expressly refers the Reader, because the object of the present Essay prescribes the greatest possible brevity, and such application could consequently not be presented in this place.

Those Readers, finally, who desire to see these views developed at length, but in such a manner as is required for pædagogical objects, will find their wishes gratified in the

"Attempt at a perfectly consequential System of Mathematics," 7 vols. (*Versuch eines vollkommen consequenten Systems der Mathematik, Berlin*),

especially in the two first volumes (2nd edition); further, but less fully, in the first vol. (2nd edition) of the

"Instruction-Book in Elementary Mathematics," 3 vols. (*Lehrbuch der Elementar-Mathematik*), which has been chiefly written for beginners;

most incompletely in the

"Shorter Instruction-Book in the whole of elementary Mathematics," 3rd edition, Leipzig, 1842 (*Lehrbuch für den gesammten mathematischen Elementar-Unterricht*), which is intended as a guide for the very earliest beginners;

on the other hand more fully and fundamentally in the

"Instruction-Book in the whole of higher Mathematics," in two volumes, Leipzig, 1839.

M. OHM.

BERLIN, *January*, 1842.

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ERRATA.

PAGE
28 line 7, *for* sect. 15, *read* sect. 16.
43 Sect. 39, line 2, *for* real *read* actual.

INTRODUCTION.

It is a remarkable fact that complaints of the want of clearness and rigour in that part of Mathematics which respects calculation,—whether it be called *Arithmetic, Universal Arithmetic, Mathematical Analysis,* or aught else,—recur from time to time, now uttered by subordinate writers, now repeated by the most distinguished of the learned. One finds contradictions in the theory of "opposed magnitudes;"—another is merely disquieted by "imaginary quantities;"—a third finally meets with difficulties in "infinite series," either because *Euler* and other distinguished mathematicians have applied them *with success* in a *divergent* form, while the complainant thinks himself convinced that their *convergence* is a fundamental condition,—or because in general investigations *general* series occur, which, precisely because they are general, can be neither accounted divergent nor convergent.

These considerations force themselves continually upon the Author of these sheets, and forced themselves more especially upon him lately when reading a letter of *Abel,* (who died so unfortunately soon for Mathematics,) which will be found in his *Complete Works* (Œuvres complètes de *N. H. Abel.* Christiania, 1839) and in which he writes thus:

"Divergent series are in general very mischievous affairs, and it is shameful that any one should have founded a demonstration upon them. You can demonstrate any thing you please by employing them, and it is they who have caused so much misfortune, and given birth to so many paradoxes. Can any thing be conceived more horrible than to declare that

$$0 = 1 - 2^n + 3^n - 4^n + 5^n - \&c. \&c.$$

when n is a whole positive number?—At last my eyes have been opened in a most striking manner, for, with the exception of the simplest cases, as for example the geometric series, there can scarcely be found in the whole of mathematics a single infinite series, whose sum has been rigorously determined; that is to say, the most important part of mathematics is without foundation. The greater part of the results are correct, that is true, but that is a most extraordinary circumstance. I am engaged in discovering the reason of this,—a most interesting problem. I do not think that you could propose to me more than a very small number of problems or theorems containing infinite series, without my being able to make well-founded objections to their demonstration. Do so, and I will answer you. Not even the Binomial Theorem has as yet been rigorously demonstrated. I have found that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1 \cdot 2} \cdot x^2 + \dots$$

for all values of x which are less than 1. When $x = +1$ the same formula holds, but only provided that m is > -1; and when $x = -1$, the formula only holds for positive values of m. For all other values of m the series $1 + mx + \dots$ is divergent. *Taylor's* Theorem, the foundation of the whole infinitesimal calculus, has no better foundation. I have only found one single rigorous demonstration of it, and that is the one given by M. *Cauchy* in his "Abstract of Lectures upon the Infinitesimal Calculus" (*Resumé des leçons sur le calcul infinitésimal*) where he has demonstrated that we shall have

$$\phi(x+a) = \phi x + a \cdot \phi' x + \frac{a^2}{2} \cdot \phi'' x + \dots$$

as long as the series is convergent; but it is usually employed without ceremony in all cases.

"The Theory of infinite series in general rests at present upon a very bad foundation. All operations are applied to them as if they were finite; but is this permissible? I think not. Where is it demonstrated that the differential of an infinite series is found by taking the differential of each term? Nothing is easier than to give examples where this rule is not correct; for example,

$$\frac{x}{2} = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \&c.$$

by differentiating we obtain

$$\tfrac{1}{2} = \cos x - \cos 2x + \cos 3x - \&c.$$

an utterly false result, for this series is divergent.

"The same remark holds for the multiplication and division of infinite series. I have begun to examine the most important rules which are (at present) esteemed to hold good in this respect, and to shew in what cases they are correct, and in what not so. This work proceeds tolerably well, and interests me infinitely."

Thus laments *Abel*. But *d'Alembert* (in many places of his "Opuscules"), *Carnot*, even *Laplace* make similar complaints, although more briefly, and not always precisely concerning infinite series. *Kramp* in his "Analysis of astronomical and terrestrial refractions, 1799, Chap. III. Analysis of numerical faculties." (*Analyse des réfractions astronomique et terrestres*, 1799, *Chap. III. Analyse des facultés numériques*), encounters a formula, of which he himself says that it is incorrect, adding: "I acknowledge frankly that all the trouble which I have given myself to find out the reason of this paralogism has been hitherto ineffectual. I should be infinitely obliged to any mathematician who would point it out to me. It seems to me to belong essentially to our theory of fractional powers and logarithms of negative numbers, and that the theorem $\log(-a) = \log a + \log(-1)$, despite its appearance of extreme simplicity, is far from having been rigorously demonstrated."

In another place of the same chapter where he meets with similar contradictions, he says further: "I should not be indisposed to believe in fact, that every application which we have hitherto made of our general theory of powers to roots and logarithms of negative quantities, were a conclusion *à particulari ad universale*, which ought not to be excusable in analysis."

Further on he says: "As regards the differential of $(-1)^x$ there is not a single mathematician, who, conforming himself to received ideas, can tell us what it is.

"But what shall we do with $(-1)^{\sqrt{2}}$? what will $(-1)^x$ become for the infinite number of cases in which the exponents are irrational, exponential, circular, or finally transcendental quantities of any description?"

Thus *Kramp*. It is clear that he complains less of infinite series and more of powers and logarithms, and, thinking that he was operating correctly, he has obtained a great number of incorrect results, but has also happily remarked that they were incorrect. On the other hand the greatest analysts, as *Euler* and *Lagrange*, have sometimes exhibited such results in their writings, without having always remarked that they were incorrect. For example, *all* the results which are to be found in the XIth Lecture of the "Lectures upon the Calculus of Functions," (*Leçons sur le Calcul des Fonctions*, 1806.) by *Lagrange*, are incorrect, and yet these incorrect results had been previously given in the same form by *Euler;* their incorrectness for a *general* and not merely *whole* exponent surprises us in *Lagrange's* work the more, that the object of the above named Lecture was precisely to prove the *general* correctness of these formulæ with perfect rigour.

But if on the one side such facts speak loudly enough, and on the other we hear complaints from men who stand on the highest height of science, and who have themselves more or less extended its bounds, we cannot but repeatedly inquire:

1. Are these complaints just or unjust, and how far so?

2. Are, as *Abel* appears to complain, the infinite series the only cause of *all* the paradoxes of calculation, or have we to seek for their sources elsewhere also, and where?

3. When we say in mathematical analysis: "this or that result is correct or is false," what do we mean by so saying?—or in other words; if two results contradict one another, what characteristic have we, which will help us to distinguish the correct from the false?

4. How may the paradoxes of calculation be avoided with certainty?

And so forth.

When the Author of these sheets considers these questions from his own point of sight, it appears to him, in examining the first question, that the reproach made by *Abel*, that *calculations* or *demonstrations* are conducted with *divergent series*, only affects in all its generality the mathematicians of the last century, since all living mathematicians of note, as *Gauss*, *Dirichlet*, *Jacobi*, *Bessel*, *Cauchy* and the rest, do not employ them, while *Poisson* among others has spoken decidedly against their being employed. But that the series which are used, and from which deductions are drawn, ought to be always and necessarily convergent, is a circumstance of which the Author of this Essay has not been at all able to convince himself; *on the contrary, it is his opinion that series, as long as they are* GENERAL, *so that we cannot speak of their convergence or divergence, must always, when properly treated, necessarily and unconditionally produce correct results.* An examination of this opinion is one of the objects of the present pages.

With regard to the second question, namely, where are the sources of the parodoxes of calculation to be looked for? the Author finds them especially: (*a*) in a onesided conception of the idea of *zero;* (*b*) in a disregard of the properties of *many*-meaning or *infinitely multiple*-meaning expressions; (*c*) in the circumstance, that through a want of proper attention in mathematical analysis, and algebra, *general* propositions are liable to be generally converted, which, as is well known, offends against the first rules of logic, and must necessarily lead to the most erroneous results; (*d*) in an imperfectly correct method of treating infinite series; and finally, also (*e*) in the fact, that results are considered and applied as generally correct, which have only been demonstrated for particular cases, and which, too, are only true in those particular cases. This last case namely occurs in the above cited work of *Kramp*, who has applied for all faculties (factorials) the law

$$h^m.\ a^{m|v} = (ha)^{m|h|v}$$

viz. for those factorials also, in which the exponent m is a broken number, whereas it is precisely this formula, among those which are employed for calculating with faculties (factorials), which does *not* hold for *broken* factorials, as may be easily proved. This is exactly the same error as would be committed if we were to pronounce that the equation

$$(-1)^n = \cos n\pi,$$

which is correct for any *whole* number n, would also hold for any broken n. For $n=\frac{1}{2}$, for example, it would give

$$(-1)^{\frac{1}{2}} = \cos \tfrac{1}{2}\pi, \text{ that is } \sqrt{-1} = 0.$$

But although even not long ago *Tralles* in two Essays of the

Academy of Sciences at Berlin, (for the years 1815 and 1821) really drew this conclusion, and seriously asserted that $\sqrt{-1}=0$; yet such confusion of ideas is no longer to be dreaded when we consider the better spirit in which mathematical analysis is at present conducted*); and hence we shall make no further reference in this place to the cause of paradoxes alluded to above, in (*e*).

We said on the other hand: (*a*) that zero is frequently conceived in too one-sided a manner. If we consider zero, namely, as the passage from positive to negative, we may, if we are compelled, say that: "$\frac{1}{0}$ is infinity," because we may then take the phrase to mean that $\frac{1}{a}$ approaches infinity the more nearly, the more nearly a approaches zero. But the case must then occur in which we cannot exactly tell whether $\frac{1}{0}$ is the positive or the negative infinity, because $\frac{1}{0}$ is also the limit of the value of $\frac{1}{-a}$ when a continually approaches zero. But because zero, much more generally, represents the difference $a-b$ for the particular case of a and b being equal, where they may be either actual or imaginary,—zero is also frequently the passage from actual to imaginary, or even from imaginary to imaginary, so that in this case $\frac{1}{0}$ would appear all at once as an infinite value intervening between values, which were all imaginary, supposing we wished to retain the same views as have been hitherto customary. Whereas a more exact knowledge concerning zero leads us, as we shall shew further on, to perceive that the form $\frac{1}{0}$ should not be allowed to occur in calculation at all, and that the introduction of this inappropriate form, is entirely without sufficient motive.

We said further, that: (*b*) the disregard of the peculiarity of many-meaning expressions led to paradoxes of calculation, and we here add, that it is *this* cause which is predominant in the above cited XIth Lecture of *Lagrange's Leçons sur le Calcul des Fonctions*. For if, for example, A were a many-meaning expression, which represents *one* of its *different* values, but which may every time that it occurs in the same calculation represent *another* one of its values, we cannot transform

$$pA+qA \text{ into } (p+q)A,$$
$$\text{nor } A^2 \text{ into } AA,$$

* It would be impossible for us to consider such errors as e.g. *Oettinger* of Freiburg (Heidelberg) has here and there committed in his writings, since they belong entirely to the man himself, and could not be easily committed a second time by any other analyst.

nor from $B = A$ and $C = A$, conclude that $B = C$,

because these conclusions would only be correct provided we had previously ascertained that A in each case represented *one and the same* of its values.

Example. If we put $7\sqrt{9}$ for $2\sqrt{9}+5\sqrt{9}$, whilst the factor $\sqrt{9}$ in $2\sqrt{9}$ represented its value $+3$, and in $5\sqrt{9}$ its value -3, we should have substituted $\pm 7 . 3$ for $2 . 3 + 5(-3)$, which would in any case be incorrect. In the same way, although $(\sqrt{4})^2$ is unconditionally $= 4$, $\sqrt{4} . \sqrt{4}$ is not necessarily $= 4$; because the factors of the product $\sqrt{4} . \sqrt{4}$ may perhaps be unequal, one representing $+2$, and the other -2, so that we should have -4 as the correct value of the product $\sqrt{4} . \sqrt{4}$.

In the above cited investigations of *Euler* and *Lagrange*, namely, the following conclusions are drawn:

$$(\cos x + \sqrt{-1} . \sin x)^m = \cos mx + \sqrt{-1} . \sin mx,$$
$$(\cos x - \sqrt{-1} . \sin x)^m = \cos mx - \sqrt{-1} . \sin mx;$$

therefore

$$\cos mx = \tfrac{1}{2}\{(\cos x + \sqrt{-1} . \sin x)^m + (\cos x - \sqrt{-1} \sin x)^m\},$$
$$\sin mx = \frac{1}{2\sqrt{-1}}\{(\cos x + \sqrt{-1} . \sin x)^m - (\cos x - \sqrt{-1} \sin x)^m\}.$$

Now as long as m is a positive whole number, the conclusions derived from these equations present no difficulty; but if we substitute a broken number for m, as $\frac{1}{3}$, then the powers on the right side of the above formulæ represent one of their several (three) values, and if we neglect, as *Euler* and *Lagrange* did, to seek out the *corresponding* values of these powers, then the above, and all similar formulæ, will invariably give false results.

It is this ambiguity of the expressions which *prohibits* us from putting $\sqrt{-a} . \sqrt{-a} = (\sqrt{-a})^2$ (that is $= -a$) generally,—and which further does *not* permit us to take $\sqrt{-a} . \sqrt{-b} . = -\sqrt{ab}$, although each may be correct in many cases. Generally, before any secondary investigations are commenced, we must take

$$\sqrt{-a} . \sqrt{-a} = \sqrt{+a^2} = \pm a,$$
$$\text{and } \sqrt{-a} . \sqrt{-b} = \sqrt{+ab} = \pm\sqrt{ab},$$

because the expressions on the right will then have just as many values as those on the left; so that none of the values on the left (among which perhaps may be exactly the one wanted in any particular application) shall be lost. For this reason it is convenient, since we cannot avoid calculating with imaginary expressions, to represent *one* of the two values of $\sqrt{-1}$ by i, and *the other* by $-i$, and to conduct the calculations so, that i, wherever it occurs, shall always represent *one and the same* of the two forms $+\sqrt{-1}$ and $-\sqrt{-1}$. By this means we avoid every difficulty, and we have, as we were used to have in elementary calculation with letters, $i . i = i^2 = -1$. But as long as we have expressions which are perfectly general, and which may be imaginary just as well as actual, it will be impossible to make such an arrangement of calculation.

We declared further (*c*) that one of the sources of paradoxes in calculation lay in the circumstance that general propositions were so frequently and unawares generally converted. This fault is most frequently committed in lower algebra, although errors in that part of mathematics are less remarked, because the investigations are there chiefly of a special character, and hence the results, before being employed, are as it were subjected to a test. One example may suffice for all. Whenever a magnitude is required in algebra, we can only reason thus: "If such a magnitude exists, and is therefore expressible by a whole or broken number x, then this number x has, by the conditions of the problem, to satisfy an equation, e.g. the equation

$$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}-1=x."$$

But we may not now say conversely: "*every* value of x which satisfies this equation, must necessarily solve the problem." For if we allowed such a conclusion, it would be no better than the following: "All men are mortal beings, therefore all mortal beings are men." Hence whenever an algebraical problem has been reduced to an equation (*stated*), we can only reason thus: "Among the values of x which satisfy this equation, that one which solves the given problem, must be included, *provided such a one exist at all.*" For the equation only expresses a single property of the magnitude sought, whereas the problem has generally one or more (secondary) conditions, which are not regarded in the equation, but which nevertheless exist, so that the required numeric value of x has not only to satisfy the equation, but also these other conditions.

Thus e.g. if x represents the radius of a circle, the secondary condition is, although not expressly mentioned, that x must be a *positive* whole or broken number. Hence if the equation gives a single value only for x, and that imaginary or negative, the problem is in either case, the latter as well as the former, in so many words, impossible, and the view which was formerly so customary, that we have only to take the negative result in the directly opposite sense in order to obtain a solution of the problem, must be distinguished as most erroneous and unfounded.

Finally that (*d*) we may not calculate with divergent infinite series (e.g. not with

$$1^n-2^n+3^n-4^n+5^n-\&c.\ \&c.$$

as long as n is positive) is at present so well understood, that we will not say any thing further upon that subject. But we shall shew in the course of the present Essay that we *can, may*, and *must* calculate, in perfect safety, with *general* series, which may be either convergent or divergent, and which are therefore neither the one nor the other.

The *third* of the above questions appears to us especially important. By what do we recognize the incorrectness of any result? All results of mathematics, namely, are *equations*;—now we cannot determine whether these are correct or not, until we exactly know what an "equation" is. Have we then in our mathematical analysis a well-defined idea of equation? The old one: "agreement or coincidence in quantity" is far too particular, and certainly no longer suitable for *those* equations in which imaginary expressions occur, and therefore much less for *general* equations, in which we do not yet know whether the expressions to the right and left of the sign (=) are actual or imaginary. But if, as some mathematicians of more modern times appear to think, we consider an equation between imaginary expressions as a symbol which combines *two* such equations between actual expressions, then we have clearly only an explanation of equations of the form $p + q.\sqrt{-1} = \alpha + \beta.\sqrt{-1}$, in which we have also the condition that p, q, α, β, are no longer general but actual. *Upon this supposition* this equation then certainly splits into two equations $p = \alpha$ and $q = \beta$. But does this last view explain the equation

$$\frac{23 + 2\sqrt{-1}}{4 - 5\sqrt{-1}} = 2 + 3\sqrt{-1},$$

which is nevertheless considered correct in analysis? Should we not be obliged *first* to substitute $2 + 3\sqrt{-1}$ for the expression $\frac{23 + 2\sqrt{-1}}{4 - 5\sqrt{-1}}$ on the left, in order to obtain the equation

$$2 + 3\sqrt{-1} = 2 + 3\sqrt{-1},$$

which is of the above form, and which would then split into the *two* equations

$$2 = 2 \text{ and } 3 = 3?$$

But that we may be enabled to substitute $2 + 3\sqrt{-1}$ for the quotient $\frac{23 + 2\sqrt{-1}}{4 - 5\sqrt{-1}}$, we must *first* know that these two expressions *may* be substituted for one another, i. e. that they are "equal;" i. e. we must *first* know what "equal" means, before we can arrive at the already mentioned view (which is rather the enuntiation of a single characteristic, than of a *definition*,) of an "equation between imaginary expressions." Moreover, we have frequently equations in mathematical analysis, between *general* expressions, which are neither actual nor imaginary, since they may be one just as well as the other. And yet we want to know whereby the correctness of such *general* equations may be recognized.

The correctness of the above particular equation,

$$\frac{23 + 2\sqrt{-1}}{4 - 5\sqrt{-1}} = 2+3\sqrt{-1},$$

is generally demonstrated à posteriori, by *multiplying* $2+3\sqrt{-1}$ by the divisor $4-5\sqrt{-1}$, and shewing that the dividend $23+2\sqrt{-1}$ is obtained as the result of this multiplication. We cannot but approve of this process, but we ask, in what "*general* idea of equation" is this process generally contained or justified? We have also the equation

$$\sqrt{-5-12\sqrt{-1}}=2-3\sqrt{-1},$$

and we should shew its correctness by squaring $2-3\sqrt{-1}$ and proving that the result will be precisely the radicand $-5-12\sqrt{-1}$ of the square root. But we must again inquire: where is the *general* idea of equation established, by which we can test *every* given equation, and shew whether it is or is not correct? Let us proceed a step further. We *multiplied* $2+3\sqrt{-1}$ by $4-5\sqrt{-1}$ above; but have we in our mathematical analysis a general idea of "multiplication," so that we can use it as a test of the correctness of our multiplication? What do we mean by saying "a is to be multiplied by b"? The *most general* definition of multiplication which can be found in Instruction-Books is: "the product ab is to be generated from the multiplicand a, in the same manner as the multiplier b has been generated from 1." And in fact we obtain from this definition not merely $4 \cdot 3 = 12$, and not merely $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$, but also $3 \cdot (-2) = -6$, $(-3) \cdot (-2) = +6$;

$$\frac{2}{3} \cdot \left(-\frac{4}{5}\right) = -\frac{8}{15}; \ \left(-\frac{2}{3}\right) \cdot \left(-\frac{4}{5}\right) = +\frac{8}{15},$$

which in the practical experiments hitherto made in the applications of calculation, are in all cases correct. But if we wished to apply the same idea of multiplication to the case in which $\sqrt{4}$ had to be multiplied by $\sqrt{9}$, we should be obliged, (since the multiplier $\sqrt{9}$ has arisen from taking 1, 9 times, and then taking the square root,) in order to obtain $\sqrt{4} \cdot \sqrt{9}$, first to take $\sqrt{4}$, 9 times, and then to take the square root of the result. This would give $\sqrt{4} \cdot \sqrt{9} = \sqrt{(9 \cdot \sqrt{4})} = \sqrt{\pm 18}$, which is absolutely false. If to avoid this contradiction it were said: "a symbolized root is not yet any quantity, hence we must take the quantities represented by the roots and multiply them together," we would propose another example, viz. to multiply $\sqrt{-4}$ by $\sqrt{-9}$, where it is not very possible to recur to the quantities represented by the roots, while the above idea of multiplication would now give

$$\sqrt{-4} \cdot \sqrt{-9} = \sqrt{(-9 \cdot \sqrt{-4})},$$

which is also absolutely incorrect, inasmuch as the further consequences deduced from this result contain contradictions. In the same way the above cited definition of multiplication would give

$$\sqrt{-1} \cdot \sqrt{-1} = \sqrt{-\sqrt{-1}} = \sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{2}} \cdot \sqrt{-1},$$

which is also absolutely incorrect. And suppose it were possible to avoid this example by some means or other, at what should we arrive?—clearly, at last, at what we desire; viz. at a *general* idea of "multiplication," of which, according to the views hitherto handed down, we have at present a total lack. But cannot similar questions be proposed respecting Powers and Logarithms?

The *fourth* question: how may the paradoxes of calculation be most securely avoided?—obliges us to submit to a very exact examination the *subject* of mathematical analysis, its *first* and *simplest* ideas, as also the *methods of reasoning* which are applied in it. This is what the author of these sheets zealously did from 1811 to 1821, and he published the *commencement* of his investigations in 1816, and only six years afterwards, a very happy result of them in the first edition (1822) of the two first volumes of his "Attempt at a perfectly consequential System of Mathematics."

And although this last-mentioned work was not received without approbation,—although its sale has enabled him to publish five following volumes of the same work,—although his shorter Instruction-Books are already spread abroad in two or more editions,—yet he can very well recollect the time when on his first appearance he hardly escaped being declared "insane" by several mathematicians on account of his views; when he was even designated in official papers as a *dangerous* innovator on account of these *revolutionary* ideas of science, and most persons contented themselves either with a silent shrug of the shoulders, or with publicly accusing him of "presumption." This public resistance only drove the author to test and retest his views continually, and if possible more rigorously, whereby his works have received a better finish, and are better adapted to satisfy the demands of scientific unity, and yet he has been unable to discover that his (certainly revolutionary) fundamental view of the subject admitted of or required any essential alteration, since by *it*, *all* formerly observed contradictions are most harmoniously solved, or, more properly speaking, *are not encountered*, and can only appear as solved or avoided, when this new process is compared with the older one, which the author terms "that hitherto employed."

The author is at this moment convinced that he has only to let his Instruction-Books work on quietly and peaceably, in order to see his views adopted by most teachers, because those books are also distinguished (precisely on account of their predominant scientific unity) by great simplicity and didactical convenience. But inasmuch as professed mathematicians, as e. g. *Abel* was, do not generally occupy themselves with reading elementary instruction-books,—*even when these would dry up the sources of their complaints*, the author endeavours in these pages to exhibit his views to such persons, in as short and comprehensive a manner as possible, and at the same time to point out the most important conclusions respecting a well assured and necessarily correct method of working with infinite series.

Now the author is convinced that all the difficulties which are met with in mathematical analysis, are to be attributed wholly and solely to the *very first* fundamental view, that view namely which each has taken for himself of the *subject* and *nature* of mathematical analysis. It appears to him namely as if the *object* had been confounded with the *means* which must be applied in order to attain that object. The object of *war* is—*peace;* but how unsuitable would it be, and to what erroneous consequences would it lead, if we declared too generally: "war is the *doctrine of peace,*" or, "war is occupied with what is peaceful"! Thus the *object* of mathematical analysis is perhaps in all cases nothing more than the *comparison of magnitudes,* but it is totally repugnant to the views of the author to say: "Mathematics," (and therefore mathematical analysis as a portion of the same,) "is the *doctrine of magnitudes* (*quantities*)." On the contrary, the author has found himself forced to conceive the nature of mathematical analysis much more abstractly, and he believes that he is much nearer the truth in asserting that: "mathematical analysis is the doctrine of the relation of those (seven) (mental) acts to one another, to which we are led by the consideration of (whole, indenominate) number," i.e. therefore "the doctrine of the oppositions" and relations (combinations) in which the above named mental operations stand to (with) one another*.

Viewed from this point of sight, the forms $a+b$, $a-b$, $a.b$, $\frac{a}{b}$, a^b, $\sqrt[b]{a}$, $\overset{b}{\log}\, a$ do *not* represent *magnitudes* (quantities), but *mental acts* (in systematic language: "symbolized operations"), which stand in certain relations to one another, that are enunciated in "equations." Every "equation" is a so-called identical one, and every such equation *never expresses anything but the relation of the operations to one another,* so that every new equation expresses the same relation, only in a new modification†. This is not only true for every equation between letters, but also for every equation between cyphers, as e.g. for the equation $65+24=89$, only that in this equation the signification of the several cyphers has been attended to, so that it no longer appears in its original purity, as it would do if written thus,

$$65+24=(60+20)+(5+4).$$

* This is curiously enough the oldest as well as the newest view, the Arabic phrase from which our word *Algebra* is derived being "el-gebr wa-l-mu*k*abalah," which may be aptly rendered "combination and opposition." *Trans.*

† So called *algebraical* and *transcendental* equations (which the author has designated *determining equations*) can only receive meaning and authority as *identical* equations; and they do *not* differ *at all* from identical equations in their nature, but only in so much that one or more of their letters has to represent determinate (generally, unknown) expressions, which must be mentally substituted for these letters.

If we now consider these equations in their simplest form, as

$$a-(b-c)=a-b+c;\ \frac{a-b}{c}=\frac{a}{c}-\frac{b}{c};\ (a+b)c=ac+bc;\ \&c.\ \&c.$$

we have the simplest laws which govern the operations (i. e. these mental acts); and the application of these laws to the formation of new and more complex "equations" constitutes "calculation," *in which idea all and every kind of calculation, the commonest as well as the most advanced, is included.*

There are only (indenominate) *whole* numbers; whatever else appears in practical calculation under the name of *number* or *magnitude* (quantity) (negative, broken, imaginary) is nothing but a combination of two or more whole numbers with one another, made by means of the above named mental operations, i. e. by means of symbolized operations. But if we start originally from whole indenominate numbers, the difference $a-b$ may be either reduced to a whole number or, it remains *per se*, and in this last case it gives the idea $b-b$ or *zero* (0), or the idea $0-(b-a)$ or $0-c$ or $-c$, (since it is usual to understand zero as the minuend of the difference, and not to write it down). We may also conceive the form $0+c$, which is usually written $+c$. But $+c$ and $-c$ and 0 are far from representing "*magnitudes*" (quantities); on the contrary, they only express the existence of combinations of numbers (i. e. of mental acts), which follow determinate laws, so that we are precisely for this reason enabled to *calculate* with these expressions.

In the same way the quotient $\frac{a}{b}$ or $\frac{\alpha-\beta}{\gamma-\delta}$ may be reduced to a positive or negative *whole* number or zero, or, it remains *per se;* in the latter case arise the ideas of *broken* number, and of *positive* and *negative broken* number. These broken numbers are therefore according to this view *not* "magnitudes" (quantities), but expressions which announce the existence of mental operations that proceed according to determinate laws, so that we are enabled to *calculate* with these expressions.

The root $\sqrt{a}$ also allows of being reduced to a prior form (i. e. to a positive or negative whole or broken number or zero), or, it remains *per se*. But in the last case it always admits of being reduced to the simplest root *per se*, viz. to $\sqrt{-1}$, so that all expressions which contain roots *per se* receive the form $p+q\sqrt{-1}$. Finally the logarithm $\overset{b}{\log}\,a$ (in especial) never remains *per se*, but may be always reduced to an *actual* number (i. e. to a positive or negative whole or broken number or zero), or at any rate to an *imaginary* number, i. e. to the form $p+q\,.\sqrt{-1}$.

According to this view the SO CALLED ACTUAL NUMBERS *are as far from being "magnitudes" as are* THE IMAGINARY. *The actual and imaginary numbers stand here in the same category; they are both of them nothing but forms* per se, *i. e. symbolized operations,* *i. e. conceived and therefore real combinations of numbers effected by means of the above mentioned mental acts, i. e. expressions which embody these mental acts, while these last are governed by determinate laws which are enunciated in equations, so that these laws can be applied, i. e. so that we can "CALCULATE" with these expressions.*

After having pointed out these general views we have now to shew that these abstract ideas possess a firm supporter. First of all it must be remembered that according to this view the *form* of an expression is at the same time its *essence*. *Form* lost, *all* lost. Hence the *form* of the expression is the supporter of the idea, and the *properties*, which this form is obliged to possess, are the *characteristics* of the idea. Finally, the relation of the operations to one another is enunciated in the determination of what new *form* is to be substituted for an old, given form; hence the whole of mathematical analysis is solely employed in the *transformation* of given forms. Consequently it is not *magnitudes* (quantities), but *forms* which are the subject of mathematical analysis.

Hence we explain a *sum* to be an expression of the form $a+b$ or $a+b+c$, &c. &c. endowed with the property, that its elements a and b, or a, b and c, &c. may be interchanged at pleasure, i. e. that $b+a$ may be written for $a+b$, or the new forms $a+c+b$, $b+a+c$, &c. instead of the old form $a+b+c$, without our having to expect any contradiction on the part of the laws of operation, (i. e. the laws of the mental acts which are here considered.)

Hence we explain a *difference* to be an expression of the form $a-b$, endowed with the property that $(a-b)+b$ may be interchanged with a itself.

Hence we explain a *product* to be an expression of the form $a.b$ or $a.b.c$, &c. &c. endowed with a double property, first that its elements a, b, c, &c. &c. may be interchanged at pleasure, and secondly that $(a+b).c$ may be interchanged with $ac+bc$,* without our having to fear a contradiction on the part of the laws of operation, (i. e. of the mental operations when abstracted from whole numbers).

Hence we explain a *quotient* to be an expression of the form $\frac{a}{b}$, endowed with the property that $\frac{a}{b}.b$ may be interchanged with a itself.

In the same way the *power* a^x must be explained to be an expression of this determinate form, and representing either other expressions with determinate properties, or these properties themselves *per se*.

Then we explain the *root* to be an expressson of the form $\sqrt[b]{a}$, endowed with the property that $(\sqrt[b]{a})^b$ may be interchanged with a itself.

* The second property enunciates the connection between the product and the sum. The first property is common to both product and sum.

Finally, we explain the *logarithm* to be an expression of the form $\log_b a$ endowed with the property that $b^{\log_b a}$ may be interchanged with a itself.

After establishing these most general ideas of the *sum* $a+b$, the *difference* $a-b$, the *product* $a \cdot b$, the *quotient* $\frac{a}{b}$, the *power* a^b, the *root* $\sqrt[b]{a}$, and the *logarithm* $\log_b a$, the ideas of "*addition*," "*subtraction*," "*multiplication*," "*division*," "*potentiation*," (or involution), "*radication*" (or evolution), and "*logarithmation*" (of a to, from, by b) allow of being perfectly generally established as signifying respectively "the *operations* by means of which these 7 *forms* are constructed." Objectively considered therefore these operations consist in simply *writing down* these symbols $a+b$, $a-b$, &c. &c.

But since these ideas must not be allowed to receive contradictory characteristics, we must be particularly careful with respect to the ideas of product and power, that the *double* or triple properties with which we endow them are not contradictory. Now in as much as at the time when the idea of product is established, there only occur (since we originally started from indenominate whole numbers,) the whole number or the difference *per se* $\alpha-\beta$ of two (such indenominate, whole) numbers, we can only at first establish the particular ideas of the product ab, (1) when a and b are whole numbers, and (2) when a and b are differences of whole numbers, and then prove that for these particular products no contradiction is to be expected. But at the time that the idea of *power* is established, we have already the (reducible and also the) quotient *per se* $\frac{\alpha-\beta}{\gamma-\delta}$, whose dividend and divisor are such differences of whole numbers, i.e. in other words, we have the so-called *actual* number, with which we must not stand in contradiction.

According to this view the most difficult point is to settle the idea of "equation;" for although we have already definitely enough enunciated the object of the equation, yet if in accordance with that object we were able to define it thus: "two expressions, i.e. two forms (symbolized operations) are equal to one another, when they may be unconditionally substituted for one another, without our having to fear a contradiction on the part of the laws of operation," it would still be necessary to possess well defined external characteristics by which the correctness or incorrectness of an equation might be ascertained. Hence we see in reference to "equation," as we saw a little while ago in reference to "product" and "power" the necessity arise of *arranging* these views, which have been here preliminarily collected generally together, *systematically*, so that they may form a well connected whole which will furnish the requisite conviction of correctness in every case. Now if we consider two expressions which are generally equal to one another, particularly, as in the case that determinate values in cyphers are substituted for all the letters; it may be always proved from the idea of equation, that both equal expressions represent either *one and the same* imaginary number $p+q.\sqrt{-1}$, or *one and the same actual* number.

If, in accordance with these views, the whole of mathematical analysis has never anything to do with "magnitudes," the ideas of "greater" and "less" cannot occur in it, except in a very improper signification. And such is the case. The *condition of two actual numbers* a and b, for which $a-b$ is equal to a positive number, is denoted by saying "a is greater than b;"—and when we say "a is less than b," we presuppose that a and b are actual numbers, (i.e. mere forms, mere symbolized operations compounded originally of whole numbers by means of the four first operations,) and we denote by the above phrase the condition of these actual numbers a and b, for which $a-b$ is equal (not to a positive, but) to a *negative* number.

According to this new view likewise the roots of positive numbers lead at first to the so-called *irrational* number, which is nothing but the sum of an infinite number of terms, i.e. a convergent infinite series. But mathematical analysis cannot acknowlege approximate values, and therefore considers the irrational number as a really infinite series, which, *precisely because it never terminates,* is considered as the *exact* value.

In the *applications* of mathematical analysis to the "comparison of magnitudes," this is subsequently differently treated. Every magnitude is at first represented by a *whole,* and afterwards also by a *broken* denominate number. If we pleased we might also introduce negative denominate numbers; but this is only possible in very rare cases, and it is best never done at all. The denomination or the unit is always previously determined upon for denominate numbers, and all that is required in any case is to consider the corresponding indenominate numbers, to compare them with one another, or to find them. Two magnitudes are called "equal," or one is said to be "greater" than the other, when they have both one and the same indenominate number, or when one is expressed by a "greater" indenominate number than the other (taken in the sense which has just been fixed for *actual* numbers,) upon the hypothesis, which is here always tacitly made, that the magnitudes which are to be compared with one another, are to be considered as expressed in such denominate numbers as have been referred to one and the same denomination. A very small magnitude will consequently be expressed by a very small actual positive number. If then the applications shew that a very small magnitude may be disregarded in comparison with another, we may of course *in such an application* disregard (neglect) the very small indenominate number which represents the former magnitude in comparison with the indenominate number which represents the latter; and thus far we are justified in substituting an *approximate value* for an irrational number, or generally for any given number, provided we term β an approximate value of a when $\beta - a$ is equal to a positive or negative number, which is, independently of its sign, very small.

Finally, we must distinguish in mathematical analysis, (1) the *perfectly general* doctrines, i. e. the perfectly general relation of the operations to one another, in which the significations of the several letters are left perfectly indeterminate, from (2) the more *special* investigations (to which e. g. the consideration of convergent series, definite integrals, and such like belong,) in which the letters have been already subjected to particular conditions. The former, i. e. the perfectly general doctrines, also contain the perfectly general infinite series, which, precisely because they are general, can neither be designated as divergent nor as convergent, whilst the most essential applications of mathematical analysis depend chiefly upon these *most general* doctrines, because we have so frequently to calculate with unknown expressions, whose combinations with one another and with known expressions it is generally impossible so to appreciate as to be enabled to pronounce, *at once, previously to any calculation*, either that they are actual, or that they are imaginary,—or that the series, which occur in any case, containing such combinations, are convergent or divergent. If then we had no *general* doctrines which could supply us with the conviction of being able to "calculate" correctly with such expressions, in spite of their being *general*, we should be obliged to give up such calculations themselves, that is, precisely those calculations whose object it is to discover the unknown expressions one by one. Hence, if we might not calculate with e. g. infinite series, till we had ascertained their convergence beyond a doubt, we should be obliged to give up as hopeless very many applications of calculation.

To illustrate this in some degree by one single example, let us consider *Taylor's Theorem.*

$$f_{x+h} = f_x + \mathrm{D}f_x \cdot h + \mathrm{D}^2 f_x \cdot \frac{h^2}{2!} + \mathrm{D}^3 f_x \cdot \frac{h^3}{3!} + \ldots$$

$$\text{or } f(x+h) = f(x) + f'(x) \cdot h + f''(x) \cdot \frac{h^2}{2!} + f'''(x) \cdot \frac{h^3}{3!} + \ldots$$

If those analysts were right, who assert, that this theorem may only be applied when it forms a convergent series, we should only be enabled to employ the differential coefficients $\mathrm{D}f_x$ or $f'(x)$, $\mathrm{D}^2 f_x$ or $f''(x)$, &c. upon the hypothesis that they were always *actual*. But how often do we apply equations like

$$\text{D}(\cos x + i \,.\, \sin x) = -\sin x + i \,.\, \cos x,$$

$$\text{D}^2(\cos x + i \,.\, \sin x) = -\cos x - i \,.\, \sin x,$$

or $\text{D}(e^{xi}) = i \,.\, e^{xi}$; $\text{D}^2(e^{xi}) = -e^{xi}$, where i represents $\sqrt{-1}$?

The idea of convergence may be certainly so generalized as to be applicable to series, whose several terms are imaginary; but by so doing we only encounter new theoretical difficulties without having set aside the old ones.

However, we will not enter any further upon infinite series at present, withholding our observations till our view of calculation has received a systematic foundation, in order then to consider infinite series more fundamentally according to that view. The object of this introduction has been simply to draw attention to the facts, (1) that when we calculate with unknown expressions, we are often quite unable to decide, during the calculation, whether the series which we have to deal with are convergent or not; and for the same reason are as little able to know, during the calculation, whether the expression with which we happen to be operating, is actual or not; broken or not; negative or not; whole or not; (2) that we must consequently find a method of calculating in safety with infinite series, even while perfectly general, so that we cannot speak either of their convergence or divergence; (3) *that the contradictions which are encountered in the higher calculus, may have, and do have their sources only in the very first elements*; finally, (4) that the views which the author has been publishing in a more complete form since 1822, may be sufficient, as soon as they are readily and willingly entered into, to set aside all those contradictions.

The author will now proceed, not only to enunciate these views shortly and distinctly, but to prove the same logically and systematically.

MATHEMATICAL ANALYSIS IN ITS RELATION TO A LOGICAL SYSTEM.

PART I.

THE RELATION OF THE FOUR FIRST OPERATIONS TO ONE ANOTHER.

ELEMENTARY ALGEBRA.

FIRST CHAPTER.

SECTION 1.

WE first start from (indenominate, whole) numbers. Two of them a and b, may be mentally combined in a third c in such a manner that c may have as many units as a and b together; and then if we suppose b and c are given, we can conceive the number a which when added to b reproduces c. These two acts of conception are now termed *addition* (of a to b), and *subtraction* (of b from c)*. These acts of conception, which are also termed (mental) *operations*, are represented by $(+)$ and $(-)$, and the above mentioned numbers by $a+b$ and $c-b$ respectively. These latter *forms* (and *not* the numbers which they represent) are termed respectively *sum* and *difference*. Objectively considered, *addition* and *subtraction* consist in simply *writing down* these forms $a+b$, and $c-b$.

SECTION 2.

From this it follows immediately, that *when sums and differences represent real numbers*, we may, agreeably to the ideas of addition and subtraction, interchange $a+b$ with $b+a$, as also $(a+b)+c$ with $(a+c)+b$, or with $a+(b+c)$, and finally $(c-b)+b$ with c.

If we then at first term two such *forms* or *expressions*, "*equal*," as represent one and the same (indenominate, whole) number and which may consequently be interchanged with one

* We beg leave to draw the reader's particular attention to the fact, that, in endeavouring to lay a rigorous foundation for a science, *nothing* may be taken for granted. Hence we are in *no* respect justified in assuming common cyphering, which must hereafter find its proper place in the edifice of the whole science. It may be in the mean time observed that what is there termed *addition* or *subtraction*, is, in fact, no addition or subtraction at all, but a *transformation* of e.g. the forms $24+35$ or $35-24$, which were obtained by the real addition and real subtraction, into the new forms 59 or 11.

another agreeably to the laws of the operations, and we use the sign $(=)$ in order to express this fact, we have immediately

(1) $a+b=b+a$.

(2) $(a+b)+c=(a+c)+b=a+(b+c)$;

for addition,—and

(3) $(a-b)+b=a$,

for subtraction, i. e. for its relation to addition.

SECTION 3.

If we would conceive the opposition between addition and subtraction, and generally the relation of these two mental acts to one another with perfect generality, we must first of all generalize the ideas of "sum" and "difference" and "equation," in order that we may no longer be obliged to consider the expressions as representing real (whole, indenominate) numbers. For this purpose we understand by the *sum* $a+b$, or $a+b+c$, (without further regarding the signification of the several letters) the mere *form*, endowed with the property that the *summands* which occur in it may be supposed to be placed in any order; and by the *difference* $a-b$, likewise the mere *form*, endowed with the property that $(a-b)+b$ may be every where interchanged with a*. If we further understand by *addition* and *subtraction*, nothing more than the *construction* of these *forms* $a+b$, and $a-b$, (and therefore, objectively considered, the mere *writing* of them *down*,) these last ideas are generalized at the same time with those of sum and difference.

Finally, if in order to have a more general idea of "equation" for these *general* sums and differences,—we term any two general expressions, i.e. such forms (as arise from so called *symbolized*, and therefore *conceived*, and therefore *real operations*) "*equal*" to one another, when they are interchangeable with one another agreeably to the laws of operation,—the question immediately presents itself: what is agreeable to the laws of operation? Now since the operations have been only abstracted from indenominate whole numbers, and we can consequently only come into contradiction with these whole numbers, the idea of "equation" must be such that two expressions which are *generally* acknowledged to be "equal," *must, in all those particular cases,* in which they represent whole (indenominate) numbers, also always represent *one and the same* (whole) *number.*

If then two expressions are "equal," whenever they may, without contradicting the laws of operation, be unconditionally substituted for one another, we shall be enabled to recognize the equality

* The difference $a-b$ consequently represents the property (and therefore every expression which possesses this property) that when the *subtrahend* b is added to it, the result is the *minuend* a.

of two expressions which are anywise compounded by (symbolized) addition and subtraction, by proving that there exists a third expression which when added to each of the two first expressions, produces two *sums*, that may be transformed into *one and the same expression*, either by applying the interchangings allowed by the definitions of sum and difference, or by substituting for these sums two other expressions which have already been ascertained to be equal in this sense.

SECTION 4.

From these definitions it follows:

(1) If each of two such expressions is equal to a third, they are equal to one another.

(2) When such equal expressions are added to, or subtracted from equal expressions they will produce equal expressions.

SECTION 5.

By means of these propositions of (sect. 4) we may immediately deduce from the equations of (sect. 2), which have been established as *general* in (sect. 3), viz. from

$$(\odot)\quad a+b=b+a; \qquad (\text{I})\quad (a-b)+b=a;$$

$$\text{and (1)}\quad (a+b)+c=(a+c)+b=a+(b+c),$$

a countless multitude of new equations; viz. among others

$$(\text{II})\quad (a+b)-b=a; \qquad (\text{III})\quad a-(a-b)=b;$$

$$\text{and}\quad (2)\quad (a+b)-c=(a-c)+b=a+(b-c)=a-(c-b);$$

$$(3)\quad (a-b)-c=(a-c)-b=a-(b+c).$$

If such equations were proposed synthetically as theorems, they could all be proved by adding the same expression (for which we should choose the subtrahends which occur in the above expressions) to both of the expressions which purport to be equal, and shewing that the definition of (sect. 3) is satisfied, i.e. that *one and the same* expression results in each case. We have not however to regard the signification of the several letters in any place, these last really becoming, *directly that the ideas have been thus generally conceived*, mere supporters of the operations, i.e. of the mental acts of addition and subtraction, considered in their mutual relation to one another, and *per se*.

SECTION 6.

To deduce new equations from given ones in pursuance of some given object, is termed "*calculation**."

* We direct the reader's particular attention to this perfectly general idea of "calculation." It includes *all kinds* of "calculation;" and may, for systematic

We are therefore able from henceforth to "calculate" with all sums and differences, *without paying any further regard to the signification of the several letters* (i.e. of the several supporters of the operations).

SECTION 7.

In the difference $a-b$ we may either suppose that a is *not* equal to b, or we may also suppose that a *is* equal to b. In the latter case we encounter the differences $a-a$, $b-b$, $z-z$, &c. &c., which are by the definition in (sect. 3) all equal to *one another*, and may therefore, by (sect. 4, No. 1) be all agreeably to the laws of operation, substituted for one another, and which we may and do consequently denote by one and the same symbol 0 (*zero*). Thus is *zero* introduced, and *in this sense* it can from henceforth be employed.

Zero is therefore the representative of the form $a-a$, (where a is a mere supporter of the operations, and may be replaced by any other symbol), and as this form is governed by the laws of subtraction, we are able to "*calculate*" with it in a determinately prescribed manner. Thus if we suppose that in (sect. 5, No. 2) $c=a$, or $c=b$, it results immediately, that

(1) $b+0=0+b=b$, and (2) $b-0=b$.

SECTION 8.

We shall then also be able to "*calculate*" with the forms

$0+b$, and $0-b$,

which are more usually written in an abbreviated form, thus

$+b$, and $-b$*.

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purposes, be thus verbally expressed: "Calculation" is the discovery, in pursuance of given objects, of *new forms* which we may substitute for *given forms* with the consciousness that such substitution will not be contradictory to the laws of operation.

Thus, if we have to *add* 24 and 65, we obtain first, from the above idea of *addition*, the form 24+65; *after that the business of "addition" is thus completed*, and the result of addition, viz. the *form* 24+65, obtained, "calculation" commences, which is no longer any addition, but simply a *transformation* of the form 24+65 into new forms, viz. first into the form (20+4)+(60+5), then into the form (20+60)+(4+5); next into the form 80+9 or the form 89, whilst we have during this transformation, *since it is performed in accordance with the laws of operation*, the consciousness, that this new form 89 may unconditionally, and in all places be interchanged with the old form 24+65, i.e. (in this particular case) that the new form 89 denotes no unit more nor less than was already denoted by the old form 24+65.

And as this simple example may have sufficed to shew, that common calculation with cyphers (which will not follow till somewhat later on,) is contained in the above given definition of calculation, the same may be in like manner shewn, or will rather be self-evident in all following "calculations."

* Since b is still perfectly general, and considered as a mere supporter of the operations, these expressions are not yet such as afterwards make their appearance under the name of *positive* and *negative* numbers. But these expressions $(+b, -b)$, may, if it be wished, be respectively termed *additive* and *subtractive expressions*.

For we find immediately from (sect. 5, Nos. 2 and 3), by putting 0 for b, and b for c, or similar substitutions,

(1) $+b=b$; (2) $a+(-b)=a-b$;

(3) $a-(-b)=a+b$;

(4) $(-a)+(-b)=-a-b=-(a+b)$.

SECTION 9.

From these last equations it follows further, that any expression anywise compounded by addition and subtraction, as

$$a-b+c-d-e+f-g,$$

(which will be perfectly determinate if we suppose that its several parts are to be added and subtracted in the order of reading from left to right, i.e. that it is the representative of the expression

$$[\{([(a-b)+c]-d)-e\}+f]-g)$$

may be always considered as the sum

$$(+a)+(-b)+(+c)+(-d)+(-e)+(+f)+(-g).$$

Whence follow

(1) the proposition, that the terms of any such compound expression, which we usually term an *algebraical sum*, may be arranged in any order whatever, and

(2) the rules for the *practical* addition and subtraction of such expressions, i.e. for the *transformation* of results obtained by the *real* addition and subtraction, i.e. by writing down e.g. the sum $(a-b+c)+(-m+n-p)$,

or the difference

$$(a-b+c)-(-m+n-p),$$

into expressions which have also the form of algebraical sums, viz. respectively into

$$a-b+c-m+n-p$$

$$\text{or } a-b+c+m-n+p.$$

SECTION 10.

After completing the doctrine of perfectly general sums and differences, and establishing the resulting rules of calculation, we may enter into particulars, i.e. into the consideration of what would result from supposing that every letter represented either a (whole, indenominate) number, or any arbitrary expression, which has been obtained from whole numbers by (symbolized) addition and subtraction. Now the proposition (sect. 9, No. 1.) teaches us that *every* final result can upon this supposition be reduced to the form $\alpha-\beta$, where α and β are any real (whole, indenominate) numbers perfectly independent of one another. This form $\alpha-\beta$ may then be transformed into

$0+\gamma$, that is $+\gamma$,

or into 0,

or into $0-\gamma$, that is $-\gamma$,

according as α has more units than, as many as, or fewer than β. Thus arise the *positive* and the *negative whole* number.

A positive or a negative whole number is therefore nothing but a *form*, which results from the symbolized (and therefore conceived, and hence real) addition or subtraction of a real (whole, indenominate) number, *to* or *from* zero; and we shall have to "calculate" with these forms according to the laws of operation enunciated in the above equations (sect. 5, 6).

By (sect. 8, No. 1.) the *absolute* (i. e. real,) whole number can be substituted for the positive whole number and conversely.

Hence we have in our subsequent investigations to take into consideration real or positive whole numbers, negative whole numbers and zero, and we must not come into contradiction with them. We comprehend all these three ideas, however, in the single *difference* $\alpha-\beta$ *of two whole numbers*, in order to proceed with logical certainty and at the same time conveniently.

SECOND CHAPTER.

SECTION 11.

WE rest the ideas of "*multiplication*" and "*division*" upon those of the *product* ab and the *quotient* $\frac{a}{b}$ or $a:b$. We understand namely by the words "*multiply* or *divide*" (a by b) nothing but "*construct* the *forms*" ab or $\frac{a}{b}$, hence, objectively considered, simply "write down" these forms. Hence the more we generalize the idea of the *product* ab and the *quotient* $\frac{a}{b}$, the more generally will the ideas of "multiplication" and "division" be conceived.

SECTION 12.

But in order to proceed systematically correctly from particulars, we first explain the *whole product* ab to be a *symbol of this form* (ab), in which the "*multiplicand*" a is already conceived as perfectly general, while the "*multiplier*" b represents a real whole number, and which denotes the sum $a+a+a+\ldots$ of b summands.

From this idea we immediately deduce

(1) $a(\beta+\gamma)=a\beta+a\gamma$ and (2) $a(\beta-\gamma)=a\beta-a\gamma$,

where a is perfectly general (a mere supporter), but β and γ are any real whole numbers subject to the condition (in No. 2.) that $\beta - \gamma$ also represent a real whole number.

SECTION 13.

Hereupon we define the *difference-product* ab as a *symbol of this form* (ab), in which a is considered as perfectly general, but in which b denotes *any* difference $\beta - \gamma$ of two whole numbers, and which represents the difference $a \cdot \beta - a \cdot \gamma$. This definition is chosen with reference to (sect. 12, No. 2) in order that the *difference-product* may include the *whole product* defined in (sect. 12) as a particular case.

In these definitions are included the equations

$$(1)\ a \cdot 0 = 0 \text{ and } (2)\ a(-\gamma) = -a \cdot \gamma,$$

where $-\gamma$ denotes a negative whole number: and we, therefore, now know what is the meaning of multiplying a perfectly general expression a by zero, or a negative whole number, i. e. what we may, agreeably to the laws of operation, substitute for $a \cdot 0$ and $a(-\gamma)$.

SECTION 14.

We next propose the three theorems,

$$(1)\quad ab = ba; \qquad (2)\quad (ab)c = (ac)b = a(bc);$$

$$(3)\quad (a+b)c = ac + bc = c(a+b),$$

and prove them *once*, for the case that all the letters a, b, c represent real whole numbers, and then *once again* for the case that all the letters a, b, c represent any differences of two whole numbers, as e.g. the differences $\alpha - \beta$, $\gamma - \delta$, $\mu - \nu$. (Compare System of Mathematics, Vol. I. sects. 90, 91.)

After this has been done, we can propose the (perfectly) *general product* ab, as a *symbol of this form* (ab), endowed with the property that ab is interchangeable with ba, as also abc with acb and with $a(bc)$, and finally also $(a+b)c$ with $ac+bc$, without having to fear that we shall, by so doing, contradict the laws of operation; just because we have as yet nothing but differences of whole numbers, while the permissibility of these interchanges has just been proved for those differences.

If we now continue to mean by "*equal expressions,*" with the same generality as in (sect. 3), such as may be unconditionally substituted for one another, without contradicting the laws of operation, then the equations (1—3) will still hold, although a, b, c are conceived with perfect generality as mere supporters of the operations, so that we can no longer speak of their particular signification; and these equations (1—3) merely express the relation in which the abstractly conceived operations *per se* of "multiplication" and "addition" stand to one another.

SECTION 15.

We next explain the *difference-quotient* $\frac{a}{b}$, or $a:b$ as a *symbol of this form* $\left(\frac{a}{b}\right)$, or $(a:b)$, in which it is presupposed that a and b represent any differences of whole numbers, and which represents that difference of whole numbers which when *multiplied* (by sects. 11 and 13) by the "*divisor*" b produces the "*dividend*" a. Hence

$$\text{(I.)}\ \frac{a}{b}\cdot b = a.$$

SECTION 16.

We soon convince ourselves:

(1) That the quotient $\frac{a}{b}$ has not always such a signification, i. e. that there does not always exist a difference of whole numbers which, when multiplied by the divisor b, produces the dividend a; and

(2) That in the particular case when a and b are equal to zero, there are an infinite number of differences of whole numbers, which may be represented by the quotient $\frac{a}{b}$; finally

(3) That, with the exception of the cases in (Nos. 1 and 2), the difference quotient always represents one single determinate difference of whole numbers.

(4) The assertions of (Nos. 2 and 3) may be also enunciated as follows: If each of two differences A and B of whole numbers, be multiplied by C (by sects. 11 and 13), and we find that $A \cdot C = B \cdot C$, then we must *necessarily* have $A = B$, provided that C *is not zero.*

This last result is of extreme importance for the whole of analysis. For it follows from it, that:

(5) If we multiply or divide equal expressions, representing the same difference of whole numbers, by other equal expressions, the result will always be equal expressions; *provided that we never divide by zero.*

(6) Whenever then a *general* divisor becomes equal to zero in particular cases, we must no longer consider equations which contain such a divisor as necessarily correct: i. e. *never divide by zero.*

The form $\frac{a}{0}$, whose dividend a is any expression, and whose divisor is *zero*, is consequently inadmissible in any calculation. Whenever it appears it is always a sign that, for the particular case which occasioned its appearance, the *general calculation* suffers an exception; that we are consequently unable to retain that *general* calculation in this particular case; that we must therefore institute a particular calculation for this particular case, at any rate from the point where we first divided by zero*.

SECTION 17.

Moreover, upon the supposition that all the letters denote differences of whole numbers, we immediately deduce from the equations of (sects. 14 and 15) viz. from

(⊙) $a.b=b.a$; (I.) $\frac{a}{b}.b=a$;

(1) $(a.b).c=(a.c).b$, and (4) $(a+b)c=ac+bc$;

any number of new equations, and among others the equations,

(II.) $\frac{ab}{b}=a$; (III.) $a:\frac{a}{b}=b$;

(2) $\frac{ab}{c}=\frac{a}{c}.b=a.\frac{b}{c}=a:\frac{c}{b}$;

(3) $\frac{a}{b}:c=\frac{a:c}{b}=\frac{a}{bc}$; (5) $(a-b)c=ac-bc$;

(6) $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$; (7) $\frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}$;

(8) $\frac{a}{b}\pm c=\frac{a\pm bc}{b}$; (9) $a\pm\frac{b}{c}=\frac{ac\pm b}{c}$;

(10) $\frac{A}{B}=z+\frac{A-Bz}{B}$; and also (11) $\frac{a}{b}=\frac{ac}{bc}=\frac{a:c}{b:c}$;

* For example, required the point of intersection of two straight lines represented by the equations $y=ax+b$ and $y=a'x+b'$. *If such a point of intersection exist*, its two coordinates will, when substituted for x and y, render the two equations (1) $y=ax+b$, and (2) $y=a'x+b'$ simultaneously identical; and we shall therefore find them by solving the equations algebraically. Now, by eliminating y we obtain

$$(3)\quad (a-a')x=b'-b;$$

hence, *if $a-a'$ is not equal to zero,*

$$(4)\quad x=\frac{b'-b}{a-a'}.$$

But if $a-a'=0$, that is $a=a'$, then we can no longer admit this result (4), although the result (3) still holds: the equation (3) now becomes $0=b'-b$, and contains a contradiction *unless* $b'=b$, and this contradiction shews, *that the supposition of the existence of a point of intersection* in this case involves a contradiction, i.e. that in this case no such point of intersection exists, i.e. *that the two straight lines are parallel to one another.*

Or, we have to transform the quotient $\frac{a+bx+cx^2+\ldots}{a'+b'x+c'x^2+\ldots}$ into a series proceeding according to whole powers of x, i.e. to find the coefficients P_0, P_1, P_2, &c. of the series $P_0+P_1x+P_2x^2+\ldots$ which is equal to that quotient. We obtain for the determination of P_0 the equation $a=a'.P_0$, whence $P_0=\frac{a}{a'}$ *if a' is not zero.* But if $a'=0$, then this general result can no longer be retained, and the direct particular treatment of this case then shews that in this case the series does not begin with such a term as P_0 but with the term $\frac{a}{b'x}$.

provided that none of the divisors is zero, and also that all the quotients denote differences of whole numbers.

And if these equations are synthetically proposed, they may be proved by multiplying the two expressions to the right and left of the sign of equality in each equation by some one of the divisors which occur in them, and applying the proposition (sect. 15, No. 4), or adding to both sides any expressions which are subtracted, and applying (sect. 3).

SECTION 18.

We can now conceive the idea of the quotient $\frac{a}{b}$ quite generally, meaning by it a mere symbol of the form $\left(\frac{a}{b}\right)$, endowed with the property that $\frac{a}{b}.b$ may be everywhere interchanged with a. Hence the general quotient $\frac{a}{b}$ represents the simple property, (and consequently every expression which possesses this property), that when it is multiplied by the *divisor b*, the result is the *dividend a*.

SECTION 19.

Hence if we still call two expressions which contain (symbolized) divisions, "*equal*," as in (sect. 3), when they may, agreeably to the laws of operation, be unconditionally interchanged, the question again arises, as in (sect. 3), what is agreeable to the laws of operation? And we must answer this question in a manner analogous to that in (sect. 3). Since we are at present only acquainted with differences of whole numbers, and can therefore only come into contradiction with them, the character of the equation must be such, that two expressions which are recognized as generally equal, shall in all particular cases,—as e. g. when all the letters denote differences of whole numbers,—represent *one and the same* difference of whole numbers. We shall therefore ascertain the equality of two expressions, compounded in any manner whatever by (symbolized) addition, subtraction, multiplication, and division, by shewing that there exists a third expression, *which is not equal to zero*, and such, that when each of the two first are multiplied by it (sects. 11 and 14), products result that may be made to pass into such expressions as are equal to one another according to the characteristic of (sect. 3), by applying the interchangings permitted by the definitions of product and quotient (sects. 14 and 18), or by substituting two expressions for one another which have already been ascertained to be equal.

SECTION 20.

Now from these definitions, in conjunction with those of (sect. 11), if follows immediately even for expressions which still contain divisions, that:

(1) If two expressions are equal to a third, they are equal to one another.

(2) When such equal expressions are added to, subtracted from, multiplied or divided by equal expressions, the results, *provided we never divide by zero, i. e. under the sole condition that no divisor is zero,* are also always equal expressions.

And from this, not only does the general correctness follow of all the equations already proposed in the preceding (sects.), but *general* "calculation" (according to sect. 6) with *general* forms, is, within the four operations hitherto considered, rigorously established, with *this one* exception, that general results are no longer to be permitted to hold, as soon as one of the divisors (in any particular case of application) becomes equal to zero. And at the same time we have in no case to regard the signification of the particular letters, so that it is convincingly self-evident, that, to whatever extent we continue the "calculation," (in the meaning of sect. 6), each several equation continues to enunciate nothing more nor less than the relation of these 4 operations or mental acts to one another,—each equation in its own peculiar manner.

But the relation of these 4 operations to one another is already completely enunciated in the following 7 equations, viz. in the equations

(1) $a+b=b+a$; (2) $(a+b)+c=(a+c)+b$;

(3) $(a-b)+b=a$; (4) $a.b=b.a$;

(5) $(ab)c=(ac)b$; (6) $(a+b).c=ac+bc$;

(7) $\frac{a}{b}.b=a$.

Any further equation (containing no other than these 4 operations) is to be considered as derived from these 7, so that any further equation only appears as a particular case, or as a combination of two or more of these 7 equations.

The two first of these seven equations contain the fundamental property of addition, the third contains the definition of subtraction, the fourth and fifth contain *that* fundamental property of multiplication which it has in common with addition, the sixth expresses in particular the connection between multiplication and addition, and completes the essence of multiplication, the seventh of these equations, finally, contains the definition of division*.

And whenever two "equal" expressions in any particular case denote differences of whole numbers, they will always denote one and the same difference of whole numbers; and there-

* We may also say: The first and second of the seven equations contain the character of the *sum*, the third the definition of the *difference*; the fourth and fifth enunciate that character of the *product* which it has in common with the sum, while the sixth shows the connection between the *product* and the *sum*, and thus completes the character of the product. Finally, the seventh enunciates the definition of the *quotient*; all conceived in the most general sense.

fore two such "equal" expressions will also denote one and the same real whole number, whenever both actually do represent real whole numbers at all; all of which necessarily follows from the given definition of the *general equation.*

Section 21.

If we now apply these equations to particular cases, we obtain at once from the ideas of (sects. 7 and 8)

(1) $a.0=0.a=0$; (2) $\frac{0}{a}=0$;

(3) $a.(-b)=(-b)(+a)=-ab$;

(4) $(-a)(-b)=+ab=ab$;

(5) $\frac{a}{-b}=\frac{-a}{b}=-\frac{a}{b}$;

(6) $\frac{-a}{-b}=+\frac{a}{b}$.

And from these result the well known rules for the "practical" multiplication of algebraical sums, when the proposition in (sect. 9), and the formulæ for sums are applied. This "practical" multiplication, namely, is no real multiplication, but a transformation of the product obtained by the real (i. e. symbolized) multiplication, into a new algebraical sum.

The meaning is perfectly analogous when we speak of the *division* of two algebraical sums. The real (i. e. symbolized) division is immediately performed; but the *desired* result, i. e. the transformation of the form (quotient) first obtained, into an algebraical sum, can only be effected by application of the formula $\frac{A}{B}=z+\frac{A-Bz}{B}$, (sect. 17, No. 10), which has to be repeatedly applied*.

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* For example, if we have to divide the algebraical sum $am-bm+cm-an+bn-cn+ap-bp+cp$ by the algebraical sum $-m+n-p$, we obtain at first from the definition of (sect. 11) the result

$$\frac{am-bm+cm-an+bn-cn+ap-bp+cp}{-m+n-p}.$$

After the *division* is thus completed, transformation or "calculation" commences, and this result has to be transformed according to the above cited formula. For this purpose we take as the first summand z, either $-a$, or $+b$, or $-c$ (or finally any other fourth expression, which however would be in so far inappropriate that the second summand which is always found from the first, would be not simpler, but more complex than the original expression, although in all cases correct.)

Thus the above quotient is first transformed (if we apply the formula (sect. 17, No. 10) in order to obtain the second summand) either into

the sum $-a+\frac{-bm+cm+bn-cn-bp+cp}{-m+n-p}$,

or the sum $+b+\frac{am+cm-an-cn+ap+cp}{-m+n-p}$,

or finally the sum $-c+\frac{am-bm-an+bn+ap-bp}{-m+n-p}$.

Section 22.

All the final results which are compounded of whole numbers by the four hitherto named operations, may be reduced to the form $\frac{\alpha-\beta}{\gamma-\delta}$, where $\gamma-\delta$ is *not zero*, but either positive or negative, but where $\alpha-\beta$ is zero, or positive or negative. This quotient therefore splits into 5 different forms, viz. into the forms

$$(\odot) \quad +\mu, \ -\mu, \ +\frac{\mu}{\nu}, \ -\frac{\mu}{\nu} \text{ and } 0,$$

where μ and ν are any real whole numbers, while $\frac{\mu}{\nu}$ is not equal to any whole number. The form *per se* $\frac{\mu}{\nu}$ is now termed a *broken number*, while the forms $+\frac{\mu}{\nu}$ and $-\frac{\mu}{\nu}$ are respectively called the *positive* and *negative, broken numbers*. These five forms, which are all contained in the single idea of

"a quotient $\frac{\alpha-\beta}{\gamma-\delta}$ of two differences of real numbers,"

are called *actual* numbers (in contradistinction to new forms *per se*, which are afterwards introduced by the generalisations of roots.)

A *fraction* or a *broken number* is consequently, according to this view, no magnitude, nor part of a whole, but a mere symbolized division of two whole numbers, which remains *per se*. It will hereafter be seen that the broken *denominate* number appears as part of a whole, and, like every *denominate* number, as a magnitude. But *no* "calculations" are *ever* made with *denominate* numbers, because by (sect. 6), "calculation" is only possible with *expressions*, i. e. symbolized operations. On the

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Now whether we take the first, or the second, or the third of these last expressions, the second summand may, since it is a quotient, be also transformed into a sum of two summands, of which the first is to be chosen arbitrarily, or in pursuance of some given object. Thus we obtain the six following new forms, which are equal to the above:

$$-a+b+\frac{cm-cn+cp}{-m+n-p}, \text{ or } -a-c+\frac{-bm+bn-bp}{-m+n-p},$$

$$\text{or } +b-a+\frac{cm-cn+cp}{-m+n-p}, \text{ or } +b-c+\frac{am-an+ap}{-m+n-p},$$

$$\text{or } -c-a+\frac{-bm+bn-bp}{-m+n-p}, \text{ or } -c+b+\frac{am-an+ap}{-m+n-p}.$$

In these six expressions we can again transform the third summand in each case, since it is a quotient, by the formula cited in the text, into a sum of two summands, of which the first may be chosen arbitrarily. If we select this first summand so that the results may be simplified, we find (in this example) that the second corresponding summand is in each case zero, so that we obtain these results, viz. $-a+b-c$, $-a-c+b$, $+b-a-c$, $+b-c-a$, $-c-a+b$, and $-c+b-a$, each of which may be substituted instead of the first quotient, because it is "equal" to it according to the idea of equation given in (sect. 3), and repeated in (sect. 19).

other hand we can at once "calculate" with fractions, as they have been just defined as symbolized divisions, according to the preceding (sections); so that a particular doctrine of fractions, such as has hitherto usually been given by itself, is, according to this view, as superfluous as a particular doctrine "for calculating with positive and negative" would have been in the preceding pages.

Section 23.

If a and b are actual numbers, then $a-b$ is either positive, or zero, or negative*. In the first case we say: a is "*greater*" than b, or b is "*less*" than a; in the second that $a=b$; in the third that a is less than b, or b greater than a. The symbols $a>b$ or $a<b$ consequently only express that $a-b$ is equal to a positive or negative number, which may be either whole or broken.

Hence it follows, as long as we only speak of actual numbers, that:

(1) If $a>b$, then $a \pm c > b \pm c$; while $ac \gtrless bc$, and $\frac{a}{c} \gtrless \frac{b}{c}$, according as c is $\left\{\begin{matrix}\text{positive} \\ \text{negative}\end{matrix}\right\}$.

(2) If $a>b$, and $b>c$, then $a>c$.

(3) Zero is greater than any negative number.

(4) Any positive number is greater than zero and also than any negative number.

(5) A negative number is the smaller, the greater its absolute term.

(6) Any *proper* fraction $\frac{a}{b}$ (where $a<b$) is less than 1; while every *improper* fraction $\frac{a}{b}$ (where $a>b$) is greater than 1.

(7) If two fractions have the same denominator, the smaller fraction has the smaller numerator; if they have the same numerator, the smaller fraction has the greater denominator; if they have different numerators and denominators, the smaller fraction has the smaller numerator, and at the same time the larger denominator.

(8) Any greater number has more units than a less one.

(9) All actual whole and broken numbers may be conceived as a series in which they become "*continuously*" greater from $-\infty$† through 0 up to $+\infty$, and, read in the contrary order, become "*continuously*" less: i. e. between any two fractions, or

* We ought properly to say: $a-b$ is "equal" either to a positive number, or to zero, or to a negative number; for the *form* $a-b$, precisely because it is *this* form, cannot be at the same time *another* form, but can only be transformed into another form which is "equal" to the former. Such phrases, which occur very frequently hereafter, are always to be understood with the *above* correction.

† This symbol ∞ is here always to be read *infinity*, and is intended to represent a positive number which is always considered to be greater than any determinate number however great.

between a fraction and a whole number, however little they may be supposed to differ from one another, an infinity of other fractions may be conceived as lying, all of which are greater than the less, but less than the greater one in the original pair. This may be made evident as follows. Take any fraction as $\frac{7}{3}$, then it will be "equal" to the fraction $\frac{7n}{3n}$; and consequently $\frac{7n-1}{3n}$ and $\frac{7n+1}{3n}$ are two fractions which are respectively less and greater than $\frac{7}{3}$ by $\frac{1}{3n}$, whatever whole number we take for n. But if we conceive the whole number n to be infinitely great, i. e. greater than any determinate whole number however great, the difference between two consecutive fractions of the following three fractions

$$\frac{7n-1}{3n}, \frac{7}{3}, \frac{7n+1}{3n}$$

will be infinitely small; i.e. always smaller than any determinate number however small.

This is what we mean here (or hereafter) when we say that these 3 fractions lie "*continuously*," or are "*contiguous*" to one another.

(10) Among the fractions which lie continuously to one another, by far the greater number have an infinitely great denominator, and then also a corresponding infinitely great numerator (whereas the fraction itself is by no means infinitely great, but lies between two other fractions that are near one another, and have finite numerators and denominators). These are called *irrational* numbers, and whole and broken numbers with finite numerators and denominators are then termed (in contradistinction to the irrational) *rational* numbers.

THIRD CHAPTER.

SECTION 24.

WE can now, starting again from particulars, first define the *whole power* a^b as a symbol of this form (a^b), in which the "*dignand*" a is conceived as perfectly general, and the "*exponent*" b is a positive (or absolute) whole number, and which denotes the product $a \,.\, a \,.\, a \,\ldots$ of b factors.

From these definitions it would then follow that:

(1) $a^{m+n} = a^m \,.\, a^n$; (2) $a^{m-n} = \dfrac{a^m}{a^n}$;

(3) $(ab)^m = a^m \,.\, b^m$; (4) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$;

and (5) $(a^m)^n = a^{mn}$.

Afterwards we define the *difference-power* a^b as a symbol of this form (a^b), in which a is perfectly general, and b any difference $\mu - \nu$ of two whole numbers, and which denotes the quotient $\frac{a^\mu}{a^\nu}$, *that is* the above product of $\mu - \nu$ factors when $\mu - \nu > 1$, or a itself when $\mu - \nu = 1$, or 1 when $\mu = \nu$ (or $\mu - \nu = 0$), or finally the quotient $\frac{1}{a^{\nu-\mu}}$ when $\nu > \mu$ or $\mu < \nu$, i. e. when $\mu - \nu$ is a negative whole number.

We then immediately prove that the same 5 equations (formulæ or laws) hold also for these difference powers.

Section 25.

"To construct," i. e. "write down the power a^b,"—this business may now be termed the *potentiation* (of a by b), and this idea of potentiation will be extended, *by leaving it verbally unaltered*, simultaneously with the idea of the power a^b itself.

Section 26.

Conceive each number in the continuously increasing series of all *positive* whole and broken, rational and irrational numbers, from 0 to ∞, potentiated by any positive whole number m, we shall then obtain another series which increases continuously from 0 to ∞ *. Consequently if a is any number of the latter series, there will *always* be a number of the first series, *and never more than one such*, which when potentiated by m reproduces the number a. This number is represented by $\sqrt[m]{a}$ and this symbol, i. e. this *form*, is called a *root*. Conceived in this particular case where a is zero or positive, it may be called the *positive* or *absolute root*. It consists of the *radicand* a and the *root-exponent* m, which last is always supposed to be positive whole.

There always exists then a positive number represented by the positive root $\sqrt[m]{a}$, and never more than one such number; and this will be, generally speaking, an irrational number, whose existence has been already pointed out, but which can never be exhibited, because it has an infinitely great numerator and an infinitely great denominator, or because it appears as an infinite series of terms all added together.

A whole or broken number, which only differs from the number we wish to assign, by a very little, is termed an *approximate value*. In applications to the comparison of magnitudes, we may usually employ an approximate value instead of the true unassignable value, *as must however first be proved on that occasion.*

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* This *must* and *can* be *proved*. See System of Mathematics, Vol. I. § 162, (2nd edit.)

Section 27.

For these positive roots the following truths may be immediately proved, viz.

(I.) $(\sqrt[m]{a})^m = a$; (II.) $\sqrt[m]{(a^m)} = a$;

(1) $\sqrt[m]{ab} = \sqrt[m]{a} . \sqrt[m]{b}$; (2) $\sqrt[m]{\frac{a}{b}} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}$;

(3) $\sqrt[m]{a^n} = a^{\frac{n}{m}}$; (4) $\sqrt[m]{(\sqrt[n]{a})} = \sqrt[mn]{a}$;

(5) $\sqrt[m]{(a^n)} = \sqrt[mp]{a^{np}} = \sqrt[m:p]{a^{n:p}}$;

where however it must be presupposed that the power-exponents are differences of whole numbers, the root-exponents on the other hand positive whole numbers, and finally the dignands and radicands all whole or broken positive numbers (or zero, provided no division by zero occurs), because in any other case either one side or the other of these equations will be entirely without signification.

Section 28.

But we may employ the preceding equation (3),—which was proved when we supposed $\frac{n}{m}$ positive or negative, whole or zero, and the dignand a positive,—for the purpose of introducing the *actual broken* (or rather the *actual quotient-*) *power*, i. e. the power, whose exponent is any actual number, but whose dignand is positive (whole or broken) or zero (but *never* negative). We shall, namely, understand by the "*actual power*" $a^{\frac{\mu}{\nu}}$, where a is positive or zero, ν positive whole, μ either zero, or positive, or negative whole, where consequently $\frac{\mu}{\nu}$ may be zero, or any positive or negative whole or broken number,—the positive root $\sqrt[\nu]{a^\mu}$ which is always positive, always exists, and is also always single-meaning.

And for these actual powers the correctness of the 5 laws of (sect. 24) may be proved over again, while (Nos. 3 and 5 of sect. 27) also hold when the powers are actual.

Section 29.

If we conceive a positive whole or broken number b, which is greater than 1, potentiated successively by all contiguous actual numbers (from $-\infty$ to 0, and then from 0 to $+\infty$), we shall obtain all contiguous positive numbers, viz. since $b^0 = 1$, in the first division all from the least, contiguous to zero, up to 1, and in the second division all from 1 to $+\infty$.

Conversely: if a is any number of the latter series of numbers, i. e. any positive number < 1, $= 1$, or > 1, there will always be a negative number, or zero, or a positive number, and never more than one (which will be most frequently irrational,) such, that if the positive number b, which is supposed to be > 1, is potentiated by it, the result is the number a.

Let us now understand by an *actual logarithm*, a symbol of the form $\overset{b}{\log}\, a$, in which a is any positive number, while b is any positive number > 1, and which denotes such a positive or negative number or zero, that when the "*base*" b is potentiated by it, the result will be the "*logarithmand*" a,—the actual logarithm therefore will always have a value and never more than one, and that value is negative, zero, or positive, according as the logarithmand is < 1, $= 1$, or > 1.

It is self-evident that the base b of the actual logarithm might also be $\lesssim 1$, provided it were positive. But if $b < 1$, then $\overset{b}{\log}\, a$ is exactly conversely positive, or zero, or negative, according as $a < 1$, $a = 1$, or $a > 1$. On the other hand the basis b may never be taken $= 1$.

SECTION 30.

For these actual logarithms the following formulæ may be easily proved:

(1) $\overset{e}{\log}\,(ab) = \overset{e}{\log}\, a + \overset{e}{\log}\, b$; (2) $\overset{e}{\log}\left(\frac{a}{b}\right) = \overset{e}{\log}\, a - \overset{e}{\log}\, b$;

(3) $\overset{e}{\log}\,(a^b) = b \,.\, \overset{e}{\log}\, a$; (4) $\overset{e}{\log}\,(\sqrt[b]{a}) = \dfrac{\overset{e}{\log}\, a}{b}$;

(5) $\overset{b}{\log}\, a \,.\, \overset{e}{\log}\, b = \overset{e}{\log}\, a$, and therefore also $\overset{b}{\log}\, a = \dfrac{\overset{e}{\log}\, a}{\overset{e}{\log}\, b}$,

where a, b, c, are also considered as positive.

Now although powers, roots and logarithms form an opposition in triplo*, and it is the problem of mathematical analysis to conceive and exhibit this opposition in its greatest generality, yet the establishment of the power a^b and the root $\sqrt[b]{a}$ in the particular case that a is negative, at once occasions the examination of so many questions, that it will be as well to pause awhile, and before continuing these investigations, to explore the ground we have already gained by the 4 first operations.

SECTION 31.

The first application of the general truths as yet discovered, is the "establishment of *calculation with cyphers*." Now all whole and determinate numbers are expressed as *sums* of several units, several tens, several hundreds, several thousands, several tens of thousands, hundreds of thousands, millions, &c. &c. which last we consider as known numbers, because they are

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* The sole reason why *addition* and *multiplication* only admit each of *one* indirect operation, viz. *subtraction* and *division* respectively, while *potentiation* admits of two, *radication* and *logarithmation*, is that $a+b$ is interchangeable with $b+a$, and ab with ba, but *not* a^b with b^a.

consecutive powers of *ten*, while we look upon the first ten numbers as known because we retain them in our memory in the order in which they occur, so that e.g. *seven* is defined as the sum of *six* and *one*, while *six* is in turn defined as the sum of *five* and *one*, and so on. Of course, in order to express *all* determinate numbers, such sums will also be necessary as have not exactly *all* consecutive whole numbers of *ten* as factors of their summands, since a single product, as 7 thousand, is in itself a determinate number.

This *sum* by which every determinate whole number is expressed, must be written like any other sum, i.e. any other symbolized addition. But it is usual to omit the sign (+) of addition, and also not to write *those* factors in the products of which the several summands consist, which are powers of ten, and therefore when we write

7943283,

we mean by this expression the *sum* of 7 millions and 9 hundreds of thousands, and 4 tens of thousands, and 3 thousands, and 2 hundreds and 8 tens and 3.

According to this arbitrary method of *writing* the *sums* which here occur, the summands, which are sometimes missing, must have their places supplied, in order that each cypher may remain in *that* place which gives it its proper value. For this purpose we can therefore employ *the* 0 (*zero*) *defined in* (*sect.* 7), since it has the property that $0.b=0$ and $a+0=a$.

Hence the symbol 10 will denote the *sum* of 1 ten + 0, i.e. exactly *ten*. Hence too the symbols 100, 1000, 10000, &c. will denote *sums* of 3, 4, 5, &c. summands, which however are respectively equal to *those* powers of *ten*, which we term *hundred*, *thousand*, *ten thousand*, &c.

Thus zero, which was introduced in (sect. 7) as the representative of a symbolized subtraction, allows of being also employed as the representative of a cypher in those symbols or forms, which consist of several cyphers written consecutively, and which express determinate whole numbers, and may be called *numerical numbers*.

Since such forms then, as 413, 6029, 700043, 62908047, &c. are only representatives of *sums*, i.e. of *symbolized* additions, we can "calculate" with them without trouble, and most conveniently by applying to them the theorems of algebraical sums.

Two such determinate numbers, as e.g. 413 and 6029 will have been *added* to, *subtracted* from, *multiplied* or *divided* by one another, as soon as we have written down the forms $413+6029$, $6029-413$, 6029×413, or $\frac{6029}{413}$ respectively, in precise accordance with the previous ideas, according to which the *thought*, and the embodying of the thought by writing constitute the real operation. But we are usually desirous of *transforming* these results obtained by real addition, subtraction, multiplication and division, viz. the sum $6029+413$, the difference $6029-413$, the product 6029×413, and the quotient $\frac{6029}{413}$, into similar sums arranged according to whole powers of *ten*, i. e. we wish to find sums (symbolized additions) arranged according to whole powers of ten, which are "*equal*" to the sum $6029+413$, the difference $6029-413$, the product 6029×413, and the quotient $\frac{6029}{413}$; and to effect this *transformation*, which is by (sect. 6) calculation, we must apply the laws of operation previously imparted, i. e. the equations which enunciate those laws; and thus common calculation with cyphers is shewn to be included in the general calculation defined in (sect. 6). These last *transformations* are always understood in practical calculation with cyphers, when mention is made of "addition," "subtraction," "multiplication," and "division."

This (practical) division of two whole numbers A and B is always effected by application of the formula (sect. 17, No. 10), the first summand z in that formula being replaced by the highest cypher of the highest order (of the highest power of ten) in order that the second summand afterwards found may only contain cyphers of inferior orders. The numbers will rarely "*divide out*" (*aufgehen*), but the final result will usually be a sum consisting of a whole number and a proper fraction, which sum is termed a *mixed number*.

SECTION 32.

"*Decimal fractions*" follow immediately upon the denary system, and the rules of quotients or sums will hold for them accordingly as they are considered as *quotients*, or as *sums* of products whose factors may also contain negative powers of ten.

The division of decimal fractions, if it were to be accurately completed, would lead to an infinite series, generally termed an "*irrational decimal fraction*," for which *in applications to the comparison of magnitudes* an approximate value may be substituted. In such an irrational decimal fraction the sum of n decimal places continually approaches nearer and nearer to a certain determinate, not infinitely great limit, even when n itself is supposed to be infinitely great; i. e. such an irrational decimal fraction is always what is subsequently termed a *convergent* infinite series.

SECTION 33.

To calculation with cyphers also belong

(1) The potentiation of a given whole or broken positive or negative number by a whole number, (involution).

(2) The radication of a positive whole or broken number,

by a whole number, exactly or approximately (evolution, or extraction of roots).

If we have to find $\sqrt[m]{a}$, where a is any positive number, write $\frac{a \cdot z^{mn}}{z^{mn}}$ where $z>1$, for a, and find w so that $w^m < a \cdot z^{mn}$, and

$$(w+1)^m > a \cdot z^{mn}, \text{ then we have}$$

$$\frac{w}{z^n} \text{ and } \frac{w+1}{z^n},$$

as two limits between which $\sqrt[m]{a}$ lies, and which differ from one another by $\frac{1}{z^n}$, so that this difference may be less than any determinate number however small, if n be taken indefinitely great (and we put for example $z=10$).

(3) Potentation of a positive whole or broken number by any actual number (by sect. 28).

(4) Evaluation of actual logarithms, and

(5) The application of logarithmic tables for the more easy completion of all the calculations which here occur.

Theoretically there is nothing here which stands in the way of all these practical operations, i.e. all these *transformations;* excepting that wherever infinite series occur, the completion of the process is rendered impossible by our own finite nature, so that we have usually to content ourselves with approximate values, which in *the applications of analysis to the comparison of magnitudes* may be substituted for the exact values, as will hereafter be shewn, when we come to consider magnitudes. In theory, that is in analysis, we always *conceive* the *infinite* series themselves, i.e. the *exact* values.

SECTION 34.

The application of the preceding laws of operation to the transformation of complex expressions, which either contain letters merely, or letters and cyphers intermixed, is termed "*the art of calculating with letters.*" It is subject to no further difficulty provided we do not take into consideration any but *whole, difference-, or actual powers;* any but *positive* (or as they are also called *absolute*) roots, or any but *actual* logarithms, and do not admit any divisor which is zero, or any logarithmic base which $=1$.

FOURTH CHAPTER.

SECTION 35.

WE have seen up to this point, that

(1) All our investigations only concern equations, and expressions, which are mere symbolized operations, and therefore mere forms.

(2) Any equation expresses simply the relation of the four first operations to one another, and this also holds of *those* equations even which contain powers, roots, and logarithms, because the latter 3 forms are as yet so specially conceived that we always may and must treat them as if they were the products, quotients, and positive or actual numbers which they represent; or, in other words, because powers, roots, and logarithms have been hitherto so specially conceived, that they have not yet appeared as forms *per se*.

(3) Any such equation, precisely because it only enunciates the relation of the operations to one another, holds perfectly independently of what the several letters in it may represent in particular, i. e. it holds for every value of every letter.

(4) But we may imagine equations which will not become correct (real) equations until perfectly determinate expressions have been substituted for some letters which occur in it, as x, y, z, &c.; and which therefore do not become correct (real) equations, until these determinate, frequently still unknown expressions, are considered as being represented by x, y, z, &c. Such equations are then called *determining equations*, because they are usually employed for the purpose of determining the unknown expressions which must be substituted for x, y, z, &c. in order to obtain the (real) correct, or, as it is usually termed, *identical* equation, i. e. in order to obtain the only equation which exists, and which enunciates the relation of the laws of operation to one another. The letters x, y, z, &c. which represent these determinate values, are then usually called the "*unknown* expressions." The process which must be applied, in order to obtain from a determining equation the determinate values of the unknown expression, is termed *the solution of the determining equation with respect to this unknown expression.* Determining equations are divided into *algebraical* and *transcendental*, the algebraical being those in which the unknown expression, conceived as being perfectly general, only occurs in the general combinations which have been hitherto considered. And *that* part of the mathematical analysis, which treats of determining equations and their solutions, is called *algebra*, and may also be subdivided into *lower* and *higher* (first and second parts of) algebra.

Section 36.

Now follows, without further difficulty, the solution of algebraical (determining) equations *of the first degree* with one or several unknown expressions, and in the latter case the exhibition of the several methods of elimination. Only from $ax = b$, we may not conclude that $x = \frac{b}{a}$ when $a = 0$, because (by sect. 16) we may never divide by zero. On the contrary when $a = 0$, the equation $ax = b$ becomes $0 . x = b$ that is $0 = b$, no longer contains the unknown expression x, and is either correct, or contains a contradiction, which announces the incorrectness of the hypothesis which has led to this equation. (Compare sect. 16. Note.)

We may not, namely, say that $x=\infty$ when in $ax=b$, a becomes zero. *But if b and a are positive numbers*, then $x=\frac{b}{a}$ will be the greater the smaller a becomes, and x will become infinitely great whenever a is infinitely small, provided that the letter b retains the determinate value which it at first possessed.

SECTION 37.

If we now proceed to the quadratic (determining) equation $ax^2+bx+c=0$, we see at once that the pure quadratic equation $x^2=q$, gives the solution $x=\sqrt{q}$ and therefore allows of *no general solution*, until we have introduced $\sqrt{q}$ for *a general expression* q.

But if we introduce the *square root* $\sqrt{q}$ for any general q by defining it as a symbol (as a form) endowed with the property that $(\sqrt{q})^2$ may be interchanged with q itself, it follows immediately from $x^2=(\sqrt{q})^2$ that $(x-\sqrt{q}).(x+\sqrt{q})=0$, that is $x=+\sqrt{q}$ and $x=-\sqrt{q}$; that is, there are *two* unequal and more complex *forms* $+\sqrt{q}$ (that is $0+\sqrt{q}$) and $-\sqrt{q}$ (that is $0-\sqrt{q}$) which equally possess this same property, and there are no more than these. Hence the general square root $\sqrt{q}$ represents in general either of two unequal forms, viz. $+\sqrt{q}$ and $-\sqrt{q}$, unless *in particular cases* a determinate one of them only be represented.

The conclusions from this are extremely important, viz.

(A) The general square root $\sqrt{q}$ has, when q is positive, two values $+a$ and $-a$ (where a is the value of the absolute root $\sqrt{q}$), one of which is positive and the other negative.

(B) When q is not positive and not zero, the general square root $\sqrt{q}$ remains a root *per se*, for which no previous (actual) form can be substituted. Every expression which without being general is not actual (i. e. not positive, negative, or zero) is called *imaginary*. Hence the general square root is in every particular case either an actual expression, or, if an expression *per se*, imaginary.

(C) For the *general* square roots (and therefore for the imaginary as well as for the actual) the laws of roots in (sect. 27), viz. the formulæ

(I.) $(\sqrt{a})^2=a$; (II.) $\sqrt{(a^2)}=a$;

(1) $\sqrt{ab}=\sqrt{a}.\sqrt{b}$; (2) $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$;

still hold, viz. (I. 1 and 2) perfectly unconditionally, because in (I.) both sides of the equation are single-meaning, while in (1 and 2) both sides of the equation are exactly *double*-meaning, so that one side is as complete as the other; while (II.) only holds unconditionally when written thus

$$\text{(II.)} \quad \sqrt{(a^2)} = \pm a,$$

because then the same forms stand on the right as are represented by the expression on the left.

(D) If $\sqrt{a}$ is general (i.e. either actual or imaginary,) or if $\sqrt{a}$ is actual or $\sqrt{a}$ imaginary, we may

not put $(p \pm q)\sqrt{a}$ for $p\sqrt{a} \pm q\sqrt{a}$,

nor $(\sqrt{a})^2$ or a for $\sqrt{a} \cdot \sqrt{a}$,

nor from $\sqrt{a} = \alpha$, and $\sqrt{a} = \beta$, conclude that $\alpha = \beta$, unless we have *previously* convinced ourselves that $\sqrt{a}$ whenever it occurs in one and the same expression, will also always represent one and the same of its two values. In other words: since this *form* is *double*-meaning, do not treat it when it occurs several times *as if* it were *single*-meaning, but always remember that although to outward appearance one and the same form, it may nevertheless represent in each case a *different* one of its two values*.

(E) If q is negative and $= -p$, then $\sqrt{q} = \sqrt{-p} = \sqrt{p} \cdot \sqrt{-1}$, while $\sqrt{p}$ is actual; so that every imaginary root of the form $\sqrt{-p}$ may be always reduced to the simpler imaginary root $\sqrt{-1}$.

(F) For calculating with *imaginary* (square) roots, there are of course no other laws or formulæ but those which were given for *general* (square) roots, namely those summed up in (C), which however must be applied with the proper precautionary rules in (D). Namely, we shall have

$$7 \cdot \sqrt{-1} - 2 \cdot \sqrt{-1} = 5 \cdot \sqrt{-1},$$

$$\sqrt{-1} \cdot \sqrt{-1} = (\sqrt{-1})^2 = -1,$$

only when we have been able so to arrange the calculations before hand that we are perfectly sure, that $\sqrt{-1}$, wherever it occurs, only represents one and the same of the two forms $+\sqrt{-1}$ and $-\sqrt{-1}$, which have in common with it its own essential property (viz. that when potentiated by 2 the result is -1).

In general therefore we must write, agreeably to the formulæ

$$\text{(C)} \qquad 7 \cdot \sqrt{-1} - 2\sqrt{-1} = (7 \mp 2) \cdot \sqrt{-1},$$

$$\text{and} \qquad \sqrt{-1} \cdot \sqrt{-1} = \sqrt{+1} = \pm 1.$$

SECTION 38.

The general solution of the general quadratic equation

$$ax^2 + bx + c = 0,$$

* The neglect of this simple rule is, as the history of mathematical analysis teaches us, a fruitful source of contradictions, or so-called "paradoxes of calculation."

now gives us without further difficulty

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

It gives for the unknown expression x, *two* and *no more than two* (unequal) values (i. e. forms which when substituted for x make the equation $ax^2 + bx + c = 0$ correct (identical), viz. $0 = 0$).

If we now consider this general solution in the particular case when a, b, c are no longer mere supporters of the operations, but are already considered as actual numbers (i. e. as expressions in cyphers of determinate form) then the two values of x are both actual or both imaginary, according as $b^2 - 4ac$ is positive or negative. But in this last case we may write

$$x = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} \sqrt{-1},$$

and we consequently perceive that *in this case both values of x may be reduced to the form $p + q\sqrt{-1}$ where p and q are actual.*

SECTION 39.

From this time forth then the possible particular forms of calculation are not merely real, but also imaginary, the latter being of the form $p + q\sqrt{-1}$; and in this form we can also include *all* actual expressions, because we may conceive $q = 0$. We term the *form* $p + q\sqrt{-1}$ therefore a *general numerical number.*

We shall always denote by i one and the same of the forms represented by $\sqrt{-1}$, so that $-i$ will then represent the other. If then $p + q \cdot i = r + s \cdot i$ *upon the hypothesis* that p, q, r, s are all four actual, we shall have severally

$$p = r \text{ and } q = s^*.$$

Hereupon we find that

(1) $(\alpha + \beta \cdot i) \pm (\gamma + \delta \cdot i) = (\alpha \pm \gamma) + (\beta \pm \delta) \cdot i$;

(2) $(\alpha + \beta \cdot i)(\gamma + \delta \cdot i) = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma) \cdot i$;

(3) $\dfrac{\alpha + \beta \cdot i}{\gamma + \delta \cdot i} = \dfrac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + \dfrac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2} \cdot i$;

(4) $\sqrt{\alpha + \beta \cdot i} = \pm\sqrt{\tfrac{1}{2}\alpha + \tfrac{1}{2}\sqrt{\alpha^2 + \beta^2}} \pm \sqrt{-\tfrac{1}{2}\alpha + \tfrac{1}{2}\sqrt{\alpha^2 + \beta^2}} \cdot i$,

where in (No. 4) the two exterior roots are considered as single meaning and absolute, and both the (+) or both the (−) signs must be taken when β is positive, while the opposite signs must be prefixed to the same two roots, that is (+) to one and (−) to the other, when β is negative.

* Since i would otherwise $\left(= \frac{r-p}{q-s}\right.$ or be) actual. We sometimes find in analytical writings the equations $X = X'$, and $Y = Y'$ deduced from $X + Y \cdot i = X' + Y' \cdot i$ while X, X', Y, Y' are still general. This is however always incorrect, and we must not be surprised if such a method of proceeding lead to contradictions.

It follows from this that all combinations of such actual or imaginary forms will always lead to the same forms. Even if we suppose the coefficients a, b, c in the general quadratic equation

$$ax^2 + bx + c = 0,$$

to be of the form $p + q.i$, it will immediately follow from the above results (Nos. 1—4) that the two values which would result from this quadratic equation for the unknown expression x, can and must assume the same form, i.e. *are general numerical numbers.*

As long, therefore, as we calculate with the means hitherto exhibited we can arrive at no other new particular forms *per se*, but all expressions which owe their existence to whole real numbers, i.e. all expressions in cyphers may always be reduced to the form $p + q\sqrt{-1}$ or $p + q.i$, where p and q are actual, so that they may also be equal to zero.

SECTION 40.

We can now proceed to the higher algebraical equations. We first prove in one of the well known methods—

(1) That every higher equation of the n^{th} degree with *general numerical* coefficients (i.e. of the form $p + q.i$) gives *one* value for the unknown expression x *of the same form* $P + Q.i$.

(2) That it gives for the unknown expression x, n such values, neither more nor less, which may however in particular cases be all or some equal to one another.

(3) That every whole function of x of the n^{th} degree, viz.

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + Px + Q,$$

with *general numerical* coefficients, (i.e. of the form $\alpha + \beta.i$) may be always split into a product of n factors (and this in only one manner) of the form

$$(x-a)(x-b)(x-c)(x-d) \ldots (x-p)(x-q),$$

where $a, b, c, d \ldots p, q$ are also general numerical expressions (i.e. of the form $\alpha + \beta.i$), namely, those values which when substituted for x make the whole function $= 0$.

(4) If the coefficients $A, B, C, \ldots P, Q$, are all actual, then those among the n factors which are imaginary may be arranged in pairs, so that each such pair may give the product

$$[x-(p+q.i)].[x-(p-q.i)], \text{ that is } x^2 - 2px + (p^2 + q^2).$$

The above whole function of x of the n^{th} degree has therefore none but actual simple factors, or at any rate actual simple and double factors (i.e. factors of the form $x^2 + r'x + s$ with *actual* coefficients).

The doctrine of the higher equations may now be continued at pleasure, without encountering any particular difficulties; we may now, namely, easily discover the characteristics by which we can ascertain whether some equal values exist for the unknown expressions, or whether all the values are unequal.

SECTION 41.

We are now able to introduce the general m^{th} root. By $\sqrt[m]{a}$ we mean a symbol, i. e. a *form*, endowed with the property that $(\sqrt[m]{a})^m$ may be interchanged with a itself, while a is conceived as perfectly general, and m is a positive whole number.

If we now endeavour to determine how many different, unequal, forms exist, represented by x, which have the property denoted by $\sqrt[m]{a}$, we find that we must have

$$x^m = a \text{ or } x^m - a = 0.$$

But since this is a higher equation of the m^{th} degree it gives m different values, neither more nor less, for x. Therefore $\sqrt[m]{a}$ is m meaning, and we find all these m different forms, by writing $\sqrt[m]{a} \cdot \sqrt[m]{1}$ for $\sqrt[m]{a}$, in which product we consider the first factor $\sqrt[m]{a}$ as merely single meaning, while $\sqrt[m]{1}$ represents the m values $\alpha, \alpha^2, \alpha^3, \ldots \alpha^{m-1}, \alpha^m$ or 1, where α has to be still determined*.

Hence it follows, that:

(A) From $A^m = B^m$ we cannot conclude that $A = B$, but that $A \cdot \sqrt[m]{1} = B \cdot \sqrt[m]{1}$, or at most that $A = B \cdot \sqrt[m]{1}$, i. e. that we can obtain A by multiplying B by one of the values of $\sqrt[m]{1}$, which has to be further determined.

(B) All values of $\sqrt[m]{1}$ have necessarily the form $p + q \cdot i$, which form we have termed the *general numerical* number.

(C) If a is actual or imaginary but of the form $p + q \cdot i$, all the m values of $\sqrt[m]{a}$ are actual or imaginary but of the same form $P + Q \cdot i$.

(D) For these general roots the following laws hold:

(I.) $(\sqrt[m]{a})^m = a$; (II.) $\sqrt[m]{(a^m)} = a \cdot \sqrt[m]{1}$;

(1) $\sqrt[m]{(ab)} = \sqrt[m]{a} \cdot \sqrt[m]{b}$; (2) $\sqrt[m]{(a : b)} = \sqrt[m]{a} : \sqrt[m]{b}$;

(3) $\sqrt[m]{(\sqrt[n]{a})} = \sqrt[mn]{a}$;

in such a manner, namely, that exactly as many and exactly the same values stand on both sides of these equations, as by (sect. 3) was required of a correct equation†.

In the same sense the equation

$$(4) \quad \sqrt[m]{(a^n)} = (\sqrt[m]{a})^n,$$

where n is considered as a positive or negative whole number or zero, is *not* correct, because the expression on the left has m values, while that on the left may have less‡, and will only have as many values when m and n have no common divisor. *The*

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* That if α is a value of $\sqrt[m]{1}$, then $\alpha^2, \alpha^3 \ldots\ldots \alpha^r$, where r is any whole number, must be also (the same or different) values of $\sqrt[m]{1}$, follows from the fact that

$$(\alpha^r)^m = (\alpha^m)^r = 1^r = 1.$$

† In (No. I.) there is only one value right and left. In (Nos. II. 1, 2.) there are exactly m values right and left, while in (No. 3) there are exactly mn values right and left.

‡ Thus for example $\sqrt[4]{(a^2)}$ has the 4 values $\sqrt{a}, -\sqrt{a}, +\sqrt{-a}, -\sqrt{-a}$, while $(\sqrt[4]{a})^2$ has only the two values $\sqrt{a}$ and $-\sqrt{a}$.

unskilful application of this equation (No. 4.) *therefore is to be looked upon as one of the sources of the paradoxes of calculation.*

(E) We may

$$\text{not put } (p+q)\sqrt[m]{a} \text{ for } p\,.\sqrt[m]{a}+q\,.\sqrt[m]{a},$$
$$\text{nor } (\sqrt[m]{a})^2 \text{ for } \sqrt[m]{a}\,.\sqrt[m]{a},$$

nor from $\sqrt[m]{a}=\alpha$ and $\sqrt[m]{a}=\beta$ conclude that $\alpha=\beta$, unless we have previously ascertained that the same symbol $\sqrt[m]{a}$ wherever it occurs in the same expression always represents one and the same of its m values,—a certainty which, *in perfectly general investigations*, we are, owing to the nature of the subject, usually unable to obtain. In the same way we may not substitute a for $\sqrt[m]{(a^m)}$, because the latter root denotes one of m different values which has no need to be a exactly, but which is certainly contained in $a\,.\sqrt[m]{1}$.

Generally, therefore, treat such a general m^{th} root as what it is, viz. a symbol which represents, wherever it occurs, *one* of m different forms, or all the m forms together, but leaves indeterminate *which one* is intended,—and you will never fall into contradictions, *whereas the many-meaningness* (ambiguity) *of roots is otherwise, when not properly regarded, a fruitful source of paradoxes of calculation.*

SECTION 42.

The elimination of an unknown expression from two higher algebraical equations, as also the solution of these equations only encounter practical obstacles, but no theoretical obstacles, which have to be logically overcome.

Remark 1. So called *transcendental* determining equations, as for example $a^x=b$, $a^{x\sqrt{-1}}-a^{-x\sqrt{-1}}=b$, where the unknown expression x appears in the exponent, cannot as yet possibly occur, because we have not yet had a power whose exponent is still unknown, and may therefore be also imaginary, nor even such powers as have their *dignand* general, and their exponent actual or broken.

Remark 2. The binomial theorem for whole exponents, which it is so easy to establish, viz. the proposition

$$(1+b)^x=1+\frac{x}{1}\,.\,b+\frac{x(x-1)}{1\,.\,2}\,.\,b^2+\frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}\,.\,b^3+\ldots$$

where x is supposed to be a positive whole number, has only $x+1$ terms on the right hand. But the series on the right may be conceived as proceeding *ad infinitum* because the coefficients of all the terms after the $x+1^{th}$ receive the factor zero, and are therefore themselves equal to zero. Now if we consider this series as really continued *in inf.* and at the same time the coefficient

$$\frac{x(x-1)}{1\,.\,2} \qquad \text{transformed into } -\frac{1}{2}x+\frac{1}{2}x^2,$$

and the cofficient

$$\frac{x(x-1)(x-2)}{1.2.3} \qquad \text{into } \frac{1}{3}x-\frac{1}{2}x^2+\frac{1}{6}x^3,$$

and the coefficient

$$\frac{x(x-1)(x-2)(x-3)}{1.2.3.4} \qquad \text{into } -\frac{1}{4}x+\frac{11}{24}x^2-\frac{1}{4}x^3+\frac{1}{24}x^4,$$

and in the same manner all following coefficients arranged according to powers of x, we may arrange the terms on the right according to powers of x, and we obtain

$$(1+b)^x = 1 + (b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \frac{1}{4}b^4 + \ldots \text{ in inf.}) \cdot x$$
$$+ \left(\frac{1}{2}b^2 - \frac{1}{2}b^3 + \frac{11}{24}b^4 - \ldots \text{ in inf.}\right) \cdot x^2$$
$$+ \ldots\ldots \text{ in inf.}$$

so that $(1 + b)^x$ is now transformed into an *infinite* series proceeding according to powers of x, whose coefficients are also *infinite* series proceeding according to powers of b. This equation holds for any positive number x; but because this series on the right can be conceived for a *general* x; it clearly furnishes a means of introducing a *general* power a^x or $(1 + b)^x$ (for any x), provided we have *previously* established the "theory of infinite series" in such a manner as to be able to use it as the foundation of rigorous investigations.

Thus we see ourselves, in endeavouring to conceive generally the opposition in triplo between the three last operations (potentiation, radication, and logarithmation), as well as the connection of these three last operations with the four first, which are called the elementary operations, again led, as we had been once or twice before, to infinite series, and we are consequently unable to defer a consideration of them any longer.

MATHEMATICAL ANALYSIS IN ITS RELATION TO A LOGICAL SYSTEM.

PART II.

THE RELATION OF THE THREE LAST OPERATIONS TO ONE ANOTHER AND TO THE FOUR FIRST.

ANALYSIS FINITORUM.

FIFTH CHAPTER.

SECTION 43.

IN the *whole function of x of the* n^{th} *degree,* that is in the form
$$a + bx + cx^2 + dx^3 + \ldots + px^{n-1} + qx^n,$$
we may consider the positive whole number n as great as we please; we may therefore consider it as *infinitely great,* i.e. always greater than any determinate number however great; hence the existence of "*infinite series proceeding according to whole powers of x*" is necessarily given. But this infinite series cannot be considered as existing, until the law according to which the coefficients are formed, has been determinately enunciated in infinitum.

Now since the infinite series proceeding according to whole powers of x is given simultaneously with the whole function of x, of an indeterminate degree, we not only *may* but we *must* "calculate" with *these forms,* whose terms proceed in inf. (according to a determinate *law of progression*), precisely according to the same laws as hold for whole functions of x of an indeterminate degree.

Now as a whole function of x ceases to be such, and consequently all the laws existing for it as such cease to be applicable, whenever a determinate value in cyphers is substituted for x,—in the same way the infinite series ceases to admit of treatment according to the laws of whole functions of x, whenever any determinate value in cyphers is given to the letter of progression x.

Consequently if we wish to "calculate" with infinite series proceeding according to whole powers of x, according to the same rules as hold for whole functions of x, with perfect certainty of success, we must make as the first condition that the letter of progression x remain perfectly general, and that no determinate value in cyphers whatever be conceived as represented by it; that it should be a mere supporter of the operations*.

And for this reason when we hereafter speak of a general infinite series, we shall not always add the words "proceeding according to whole powers of x or z or ϕ, &c." since it must necessarily "proceed according to whole powers," and the letter of progression does not always need to be mentioned when it cannot otherwise be mistaken.

SECTION 44.

From this we deduce the following truths, which hold for such general series proceeding according to whole powers of any letter of progression:

(1) Since a whole function of x of the n^{th} degree must have all its coefficients equal to zero, if it is to be equal to zero for all values of x, that is, while x remains perfectly general†,—in the same way a series proceeding according to whole powers of x must have nothing but zeros as its coefficients, if it is to be always equal to zero while x is quite general.

(2) Since two whole functions of x of the n^{th} degree, if they are to be equal to one another for all values of x, that is, while x remains perfectly general, must have all the coefficients of like powers of x respectively equal to one another,—similarly, this must be also necessarily the case, when two infinite series proceeding according to whole powers of x are to be equal, while their letter of progression remains perfectly indeterminate, or rather general (a mere supporter of the operations)‡.

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* It must here not be overlooked that the whole of analysis, i.e. the so called calculating part of mathematics, has *nothing* to do with magnitudes, and that we can and may *never* "calculate" with magnitudes (sect. 6), but only with forms (with symbolized operations), and that consequently all "calculation" with infinite series must necessarily cease, directly the *form* ceases with which calculation may and can be carried on.

On the other hand, after having introduced the *general* power, as is done below, we may multiply such a series proceeding according to whole powers of x by $x^{\pm\frac{\mu}{\nu}}$; so that we obtain a series which begins with negative powers of x, and proceeds according to broken powers of x. Similarly, we may substitute, for example, $\frac{1}{z}$, or z^{-1} for x, so that the series proceeds according to ascending whole powers of z^{-1}, and therefore may be made by the slightest transformation {of $(z^{-1})^{\nu}$ into $z^{-\nu}$} to proceed according to negative powers of z. But all this can also be done to whole functions of x of the n^{th} degree without in any respect altering their character.

† For if every coefficient were not zero, we should have an algebraical equation of the n^{th} or a lower degree, which would give at most n or fewer values of x, and therefore would not hold for *all* values of x.

‡ Different proofs have been given of these two propositions, but all of them have been justly attacked. If from

$$(1) \quad a_0+a_1x+a_2x^2+a_3x^3+\ldots \text{ in inf.}=0$$

we concluded that $a_0=0$, because the equation would not otherwise hold for $x=0$,

Section 45.

The practical *addition, subtraction,* and *multiplication* of two such infinite series R and S, proceeding according to whole powers of x, that is, the *transformation* of the sum $R+S$, the difference $R-S$, and the product $R.S$ into similar infinite series, presents, according to what has preceded, no difficulty at all.

But if the quotient $\frac{R}{S}$, or $R.\frac{1}{S}$, or $\frac{1}{S}$ has to be transformed into a similar infinite series, then the coefficients A_0, A_1, A_2, A_3, &c. &c. of an infinite series $A_0+A_1.x+A_2.x^2+A_3.x^3+\ldots$ have to be sought, such that the latter when multiplied by S gives R or 1; now the multiplication and comparison of the product with the series R or 1, *goes on in infinitum;* and we obtain new equations *in infinitum* between the coefficients, so that from each consecutive equation a consecutive coefficient is continually found from among the coefficients A_0, A_1, A_2, A_3, &c. which were all at first unknown, *and this in infinitum.* The new *infinite* series $A_0+A_1x+A_2x^2+A_3x^3+\ldots$ has therefore *in infinitum* the property, that when multiplied by S, the result is a new series which coincides *in infinitum* with the series R or with 1. Now since the quotient $\frac{R}{S}$ or the quotient $\frac{1}{S}$ is by (sect. 18) a mere *form,* which represents the property (and therefore every expression which possesses this property) that $\frac{R}{S}$ when multiplied by S gives R,—or that $\frac{1}{S}$ multiplied by S gives 1; and since the *infinite* series $A_0+A_1x+A_2x^2+\ldots$ just found possesses this property, it follows by (sect. 3) that it is "equal" to the quotient $\frac{R}{S}$ or $\frac{1}{S}$, and may be every where substituted in "calculation" for the quotient $\frac{R}{S}$ or $\frac{1}{S}$ unconditionally, without our having to fear that we shall thereby come into contradiction with the laws of operation*.

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whereas it ought to hold for *all* values of x, such a conclusion might be permitted. But then we have also

$$a_1x+a_2x^2+a_3x^3+\ldots=0,$$

$$\text{i.e. (2)}\quad x(a_1+a_2x+a_3x^2+\ldots)=0,$$

and *either* $x=0$, *or*

$$(3)\quad a_1+a_2x+a_3x^2+\ldots=0;$$

that is, this last equation is only *necessarily* correct when x is not zero, since for $x=0$ the equation (2) is already satisfied, without requiring the second factor $a_1+a_2x+a_3x^2+\ldots$ to be zero. That therefore $a_1=0$ *necessarily* does not follow, because the equation (3) does not necessarily hold for $x=0$.

But if we keep the view given in the four preceding chapters carefully in mind, the explanation furnished in the text presents no difficulty whatever.

* We obtain the same result if we apply the formula (sect. 17, No. 10)

$$\frac{A}{B}=z+\frac{A-Bz}{B}$$

We may add here, by way of illustration,

(1) The coefficients A_0, A_1, A_2, A_3, &c. cannot be determined when more first coefficients in the divisor S are equal to zero than in the dividend R*, in other words, when the quotient $\frac{R}{S}$ reduces to the form

$$\frac{x^m.(b_m + b_{m+1}.x^1 + b_{m+2}.x^2 + b_{m+3}.x^3 + \dots)}{x^n.(c_n + c_{n+1}.x^1 + c_{n+2}.x^2 + c_{n+3}.x^3 + \dots)},$$

while $m < n$, and b_m and c_n are not equal to zero. This is also perfectly self-evident, because the result of the transformation can be at most

$$\frac{1}{x^{n-m}}.(C_0 + C_1.x + C_2.x^2 + C_3.x^3 + \dots).$$

(2) If it has been shewn that the *infinite* series

$$A_0 + A_1.x + A_2.x^2 + A_3.x^3 + \dots$$

is "equal" to the quotient $\frac{R}{S}$ it follows as a matter of course that this is no longer the case for any *finite* number of terms of this series, however many we may take of them. Hence if we only wished to retain a number n of the first terms of the infinite series thus found, we should be obliged to subjoin a *complementary term*, i.e. we must conceive an expression E, which is usually unknown, and cannot perhaps be exhibited in a finite form, to be subjoined, so that

$$A_0 + A_1.x + A_2.x^2 + \dots + A_{n-1}.x^{n-1} + E$$

shall be again "equal" to the quotient $\frac{R}{S}$. This complementary term E is (although not in the above case, yet in the following developments) usually nothing but a symbol, which represents the sum of all the following infinitely many terms of the *infinite* series first found, and therefore itself denotes an infinite series.

(3) Since the coefficients of the infinite series R and S are arbitrary, they may, beginning from a certain term, be conceived as *all* equal to zero *in infinitum*. In this case we have transformed a broken rational function of x, as

$$\frac{1}{1+x}, \quad \frac{1}{1-x}, \quad \frac{1-x}{1+x^2},$$

$$\text{and generally} \quad \frac{b_0 + b_1 x + b_2 x^2 + \dots + b_m x^m}{c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n}$$

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continually *in infinitum*, and at the same time do not forget that A and B must here be treated as *whole functions of x*. If we were to stop in this division, we should have a *complementary term* in addition to the *finite* number of terms hitherto found, in order to obtain a correct equation; but if we conceive the division to be really *continued in infinitum*, no complementary term can ever appear.

* The general calculation would namely in this case give the form $\frac{q}{0}$ for A_0, which form does *not* shew that the coefficient is infinitely great, but which form *always* shews that the *general* method of calculation does not apply to this particular case, and that this case must be particularly treated.

into an *infinite* series which may be every where substituted for it without having to fear a contradiction on the part of the laws of operation.

SECTION 46.

Since when m is a positive whole number, the m^{th} power of a series proceeding according to powers of x, is nothing but the product R . R . R ... of m factors, there is no difficulty in the transformation of R^m, when m is a positive whole number, into a similar series. The same is also the case with R^{-m}, since $R^{-m} = \frac{1}{R^m}$, and the division has been completed in (sect. 45). On the other hand we cannot at present speak of a broken power of an infinite series, since we at present are only acquainted with such broken powers as have positive dignands (comp. sect. 28), while such a series as R is here considered as a *general* form, in which we have not to regard the signification of the several letters.

But if we have to extract the m^{th} root of the series R proceeding according to whole powers of x, i.e. if we have to transform $\sqrt[m]{R}$ into a similar series, we must find the coefficients A_0, A_1, A_2, A_3, &c. of an infinite series

$$A_0 + A_1 . x + A_2 . x^2 + A_3 . x^3 + \ldots$$

so that when this series is potentiated by the positive whole number m, the series R results. Now if we really raise this series

$$A_0 + A_1 . x + A_2 . x^2 + \ldots$$

to the m^{th} power, and compare the result with the series R, we obtain, since the coefficients of like powers of x must be equal, an infinite number of equations between these coefficients; of which each one in succession *in infinitum* will determine a consecutive one of the coefficients A_0, A_1, A_2, A_3, &c. &c. which were at first unknown. Therefore there exists an *infinite* series, $A_0 + A_1 x + A_2 x^2 + \ldots$ which when potentiated by m, gives a new infinite series, whose terms coincide *in infinitum* with those of the series R. Now since $\sqrt[m]{R}$ represents each of m expressions, which have the property of, when potentiated by m, giving the radicand R, (by sect. 41), and since the infinite series $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$ just determined has this property, we have the generally correct equation (sect. 3)

$$\sqrt[m]{R} = \sqrt[m]{1} . (A_0 + A_1 . x + A_2 . x^2 + A_3 . x^3 + \ldots),$$

in case the coefficients A_0, A_1, A_2, &c. &c. have been so chosen as to be only single-meaning for the more convenient discovery of the corresponding values.

We find, however, in actually performing the process, this exception: the coefficients A_0, A_1, A_2, &c. can *not* be determined when a certain number, which is not exactly m, or $2m$, or generally νm, of the first coefficients of R are successively $= 0$*. But this is self-evident in a direct method, because the series R can then be reduced to the form

$$x^n . (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots)$$

and an m^{th} root in the required form is only possible when $n = \nu m$, ν being a positive whole number.

The infinite series R proceeding according to whole powers of x, whose m^{th} root has been transformed into a similar infinite series, may also have all its coefficients after a certain term, equal to zero, so that it becomes nothing more than a whole function of x of the n^{th} degree. According to what has preceded then, the m^{th} root of a whole function of x, may, with the above mentioned exception, be always transformed into a series proceeding according to whole powers of x.

We can now transform an irrational broken function into an infinite series proceeding according to whole powers of x, and that in all cases, while the coefficients are still perfectly general, although the transformation will admit of exception for particular values of the coefficients, i. e. when certain coefficients are equal to zero. Namely, when R and S are whole functions of x, and therefore $\frac{R}{S}$ a broken rational function of x, we have $\sqrt[m]{(R : S)} = \sqrt[m]{R} : \sqrt[m]{S}$, where the dividend and divisor on the right, and therefore also the quotient itself can be transformed into a similar series, at least as long as the coefficients are quite general and none of them has as yet received the value zero. The same holds for other compound expressions, e. g. for $\sqrt[m]{R}$; $\sqrt[n]{S}$, &c. &c.

SECTION 47.

We can consequently always transform *any* expression anywise compounded of whole functions of x, or infinite series proceeding according to whole powers of x, as far as we yet know of the existence of any such expression (that is, provided no general power y^x and no general logarithm $\log y$ occurs, whose signification in calculation we have not yet been able to establish in these pages,) as long as the coefficients of the given infinite series or whole functions of x, are still perfectly general and have not yet received any particular value in cyphers,—we can always transform such an expression into a series proceeding according to whole powers of x, which is "equal" to it in the

* For the expressions for A_0, or A_1, or A_2, &c. take the form $\frac{q}{0}$, which form always shews that the particular case which leads to it is not contained in the *general* calculation, and that consequently the calculation must be conducted in a particular manner for this particular case. No one can in such a case seriously believe that the coefficient $\frac{q}{0}$ is infinitely great.

sense of (sect. 3), i. e. which may be unconditionally substituted for it in all "calculations" (sect. 6), without our having to fear contradictions on the part of the laws of operation.

There are therefore finite expressions which can be *transformed*, or, to employ the usual phrase, *developed* into general infinite series. From this it follows conversely that there will be some general infinite series which are "equal" (sect. 3) to a finite expression, i. e. for which a finite expression can be substituted in all "calculations" without our having to fear any contradictions on the part of the laws of operation.

This finite expression which can be substituted for such a general series, is commonly called the *sum* of the *infinite* series; and to find it, is usually termed "*to sum the series.*" Such a *sum* only exists in very rare cases, but we often transform an infinite series into an expression compounded out of other *infinite* series, and we say that the first series is *summed*, when the latter series admit of more convenient treatment, and have been represented by simple symbols (as a^x, e^x, sin x, cos x, &c.)

We can consequently *only* speak of the *summation* of an infinite series, *when it is still conceived as perfectly general*, i. e. when it proceeds according to the powers of any expression x which is still general, i. e. which is still a mere supporter of the symbols of operation, and by which we do not as yet suppose any determinate value in cyphers to be represented*.

SECTION 48.

If in an infinite series proceeding according to whole powers of x, a determinate value in cyphers be given to x, we obtain an expression consisting of an infinite number of terms (in which the terms are still considered as perfectly general, or have already received certain values in cyphers), which has no longer the form of whole functions of x, and with which we are therefore no longer able to "calculate" according to the laws of the whole functions of x, but which has still the form of an algebraical sum of an infinite number of terms (whose terms must also follow *in infinitum* a determinately enunciated law, since it cannot otherwise be considered as given *in infinitum*,) so that we can still apply the laws of algebraical sums for the purpose of calculating with them.

But if such series have to be multiplied or divided by one another, we have usually no law for arranging the terms of the result, i. e. we have no determinately prescribed *form* to which the result is to be reduced; and it is consequently a very precarious affair to "calculate" with them (according to sect. 6); and if a calculation with them succeeds, i. e. leads to a desired

* We shall soon prove the existence of *numerical* infinite series, and then shew that although they have no *sum* as has been defined in the text, yet they have a *value*, and that though we cannot *sum*, we can *evaluate* them.

application, it has usually to be ascribed solely to the circumstance that the calculation has been conducted as if the several terms of the series had been still multiplied by the successive powers of an expression x, which does not else appear in the terms themselves, that is, as if they were series which proceeded according to whole powers of x*.

If on the other hand we consider all the terms in such an infinite series which no longer proceeds according to whole powers of any expression x, as determinate values in cyphers, i.e. as perfectly determinate actual or imaginary numbers of the form $p+q\sqrt{-1}$, these infinite series may be called *numerical* series in contradistinction to the *general* series, and in these *numerical* series we must carefully distinguish two cases.

(*a*) If the result obtained by adding together n consecutive terms of the series, receives a perfectly determinate actual or imaginary (limiting) value $p+q\sqrt{-1}$ † for $n=\infty$ (i.e. when n is infinitely great, i.e. in case a positive whole number, greater than any determinate number however great, is substituted for n); in this case the (numerical) series is termed *convergent* (and we say it *converges*) and the above mentioned actual or imaginary number $p+q\sqrt{-1}$, which has resulted from putting $n=\infty$ in the result of the addition of the n first consecutive terms of the series, is then called the *value* of the convergent (numerical) series ‡;

(*b*) If the addition of n terms of the series gives an expression of the form $P_n+Q_n\sqrt{-1}$, in which however one or each of the functions P_n or Q_n becomes, for $n=\infty$, itself infinitely great (i.e. positive or negative, but absolutely greater than any determinate whole or broken number however great); such a (numerical) series is called *divergent* (and we say *it diverges*).

As long, therefore, as an infinite series is still conceived as *general*, we cannot speak about its divergence or convergence, precisely because, according to what has gone before, these latter ideas can only make their appearance in conjunction with *numerical* series (that is, when the general series are expressly considered as numerical).

From these ideas the following most important conclusions may now be drawn:

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* And herein we see the explanation of the circumstance that "calculation" with *divergent numerical* series, as may be often found in the writings of the mathematicians of the last century, may nevertheless lead to a correct result.

† The result of the addition of n such terms has namely the form $P_n+Q_n.\sqrt{-1}$, where P_n and Q_n are functions of n which receive actual values for every positive whole n, while Q_n may also be $=0$. If then P_n and Q_n receive determinate actual values p and q for $n=\infty$, the above case occurs. The numbers p and q may however be only given by finding that P_n and Q_n lie for $n=\infty$ between two limits which may be made to approach one another as near as we choose, in other words, p and q may be irrational.

‡ What is here called *value* is frequently termed by other writers *sum*. But we *here* distinguish accurately between *value* and *sum*, and only use the word *sum* in the signification attached to this word in (sect. 47).

(1) The convergent numerical series has a "value" which is actual or imaginary and of the form $p + q\sqrt{-1}$; this "value" can always be substituted for it in all "calculations," since it is "equal" to it. And *inasmuch as it always reproduces this "value" and no other than this value,* we can always "calculate" with numerical series which are also convergent.

(2) Now a convergent series has this value only by virtue of the law according to which its terms proceed *in infinitum.* Hence to prevent any doubt occurring concerning the true value of a convergent series, we must carefully enunciate the law according to which the terms are to be taken *in infinitum.*

For example, from the same reciprocal terms of the natural numbers, viz.

$$1,\ -\frac{1}{2},\ +\frac{1}{3},\ -\frac{1}{4},\ +\frac{1}{5},\ -\frac{1}{6},\ +\frac{1}{7},\ -\text{in inf.}$$

we may compose any number of numerical infinite series, which are all convergent but have all different values, each however having its own perfectly determinate value by virtue of the determinate law according to which it is constructed.

The series $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \text{in inf.}$

in which, if we take $2n$ terms, there are always as many positive as negative terms, has for its value log. nat. 2.

The series $\quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \text{in inf.}$

in which, if we take the first $3n$ terms, $2n$ positive and only n negative of the above terms follow one another (in their order) has for its value $\frac{3}{2}$ log. nat. 2.

The series $\quad 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \text{in inf.}$

in which, if we take the first $3n$ terms, we find n of them positive and $2n$ negative, has for its value $\frac{1}{2}$ log. nat. 2.

And if we take μ of the above terms positive and ν negative, and then μ positive and ν negative, and so on, the value of the resulting numerical convergent series is $=$ log. nat. $2 + \frac{1}{2}$ log. nat. $\frac{\mu}{\nu}$, and therefore $=$ log. nat. 2, which is that of the first series, when $\mu = \nu$, but greater than that when $\mu > \nu$ and less when $\mu < \nu$.

(3) A divergent series has *no* value which it can represent; a divergent (numerical) series is therefore a form inadmissible in calculation, exactly as the form $\frac{b}{0}$ was above shewn to be, and must be acknowledged as such.

Consequently, if a general series becomes a numerical one, in any particular case of application, and this numerical series is found to be divergent, we are instantly incapable of calculating with it, and the result, *without being incorrect,* shews decisively, *that the general calculation previously employed no longer holds when the letters receive these values in cyphers, and that the calculation must be recommenced and particularly conducted for this particular case* *.

* Among divergent series are those in which the same terms recur periodically *in infinitum*, as the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + 1 - \text{in inf.}$$

(4) From every numerical series R, whether convergent or divergent, a general series can be formed, by subjoining to the several terms the different powers of a general expression x, as factors. If the new series be designated by R_x, the *numerical* series R_1 will result from putting $x=1$ in the *general* series R_x.

Now suppose the general series R_x to have a "sum" (in the sense of (sect. 47) which may be represented by S_x, so that we have

$$(\odot)\ldots\ldots\ldots R_x = S_x,$$

and that this sum S_x receives for $x=1$ a determinate value in cyphers, which may be represented by S_1, then the equation ($\odot$) becomes for $x=1$,

$$(\rangle)\ldots\ldots\ldots R_1 = S_1.$$

Now if the series R_1 is convergent, then its "value" is necessarily S_1, but if the series R_1 is divergent, it has no value, and that, *which has no existence*, cannot of course pretend to be equal to the value S_1. The equation $R_1 = S_1$ ceases to exist, as far as calculation is concerned, when the series R_1 is divergent, just because the divergence of the series always shews by (No. 3), that the general calculation here suffers an exception.

This may be now more generally enunciated. If a general series R_x has generally the "sum" S_x, so that the equation

$$(\alpha)\ldots\ldots\ldots R_x = S_x$$

is a correct one (in the sense of sect. 3); and if a determinate value in cyphers is given to x, so that the series R_x becomes a numerical series R, while S_x receives the value in cyphers S, the equation

$$(\beta)\ldots\ldots\ldots R = S$$

is still correct, provided R is convergent, i. e. S is then the "value" of the infinite convergent series R; while if the numerical series R is divergent, the equation (β) does not enunciate anything which is incorrect, but it no longer holds; it ceases to exist, and can no longer be regarded in calculation*.

And still more generally. If two forms R_x and S_x, which are either both finite or of which one contains infinite series, or both contain infinite series, are equal to one another, and if R and S represent those forms expressed in cyphers, which result from substituting a determinate value in cyphers for x in R_x and S_x respectively, then the equation $R=S$ is correct, as long as R and S have determinate "values" (in cyphers); and on the other hand not incorrect, but no longer admissible in calculation, when-

* This is precisely the same case as that of the equation $\frac{b}{a}\cdot a = b$. This equation is always correct as long as a is general (that is, a mere supporter of the operations);—but if the value 0 (zero) is given to a, it does not become incorrect, but entirely ceases to exist, because a form like $\frac{b}{0}$ cannot be admitted in calculation. It always points out an exception.

ever one or each of the forms in cyphers R and S, contains a (numerical and) divergent infinite series.

Since these doctrines and rules which result necessarily from the present views, have been here and there overlooked and disregarded by analysts, we will illustrate the subjects we have just been treating, by a few examples.

Thus, for example, the "sum" of the general series

$$1-2x+4x^2-8x^3+16x^4-\text{in inf.}$$

is $\frac{1}{1+2x}$. Now whenever $x<\frac{1}{2}$ this series converges, and its value is then found from that of its sum $\frac{1}{1+2x}$ for the same value of x, so that its value for $x=\frac{1}{4}$, for example, is necessarily $=\frac{2}{3}$. But for every value of $x>\frac{1}{2}$, as e. g. for $x=1$, this same infinite series diverges, that is, has no value at all; and that which has no existence cannot of course be found from the general sum $\frac{1}{1+2x}$ by putting $x=1$. The equation

$$1-2x+4x^2-8x^3+16x^4-\text{in inf.}=\frac{1}{1+2x}$$

is therefore a perfectly correct equation. Both *forms* to the right and left of the sign of equality have the one single property which the quotient on the right represents, viz. that when they are multiplied by $1+2x$ the result is exactly $=1$. Either general expression may therefore be unconditionally substituted for the other in all "calculations." And in the case that *both* expressions to the right and left of the sign of equality receive values in cyphers, these *two* values have the same property and are therefore still "equal" to one another. But we are of course unable to say any thing at all about this latter circumstance, when one of the two values no longer exists, i. e. when the series diverges.

We may also transform the quotient $\frac{1}{1+x}$ or $\frac{1}{x+1}$ into a series proceeding according to whole powers of $\frac{1}{x}$. We have therefore the following equations which are generally correct:

(1) $\frac{1}{1+x}=1-x+x^2-x^3+x^4-\text{in inf.}$

(2) $\frac{1}{x+1}=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}+\text{in inf.}$

(3) $1-x+x^2-x^3+\text{in inf.}=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}+\text{in inf.}$

If we now transpose in this last equation some of the terms to the right from the left, or to the left from the right, we have the additional correct equations:

(4) $\frac{1}{x^2}-\frac{1}{x}+1-x+x^2-x^3+\text{in inf.}=\frac{1}{x^3}-\frac{1}{x^4}+\frac{1}{x^5}-\frac{1}{x^6}+\text{in inf.}$

(5) $-x^3+x^4-x^5+x^6-\text{in inf.}=-x^2+x-1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\text{in inf.}$

If we now write the left side of the equation (4) thus

$$\frac{1}{x^2}(1-x+x^2-x^3+\ldots)$$

and sum it by (No. 1) we obtain

(6) $\frac{1}{x^2(1+x)}=\frac{1}{x^3}-\frac{1}{x^4}+\frac{1}{x^5}-\frac{1}{x^6}\text{ in inf.}$

If we write in equation (5) the expression on the right thus,

$$-x^3\left(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}+\text{in inf.}\right)$$

and sum it by (No. 2), we obtain

(7) $-x^3+x^4-x^5+x^6-\text{in inf.}=-x^3\frac{1}{x+1}=-\frac{x^3}{x+1},$

and these two last equations are also unconditionally correct (by sect. 3).

But if we now give determinate values in cyphers to x, then the infinite series are no longer general, but numerical, and the equations without becoming incorrect, either cease to exist, or become correct equations in cyphers.

(No. 1) viz. gives on the left the value of the numerical convergent series on the left when $x<1$; when $x>1$ the equation (No. 1) ceases to exist, because the series then diverges. The first is the case in (No. 2) when $x>1$, the second when $x<1$. The equation (No. 3) although it is entirely and perfectly correct, and although we have deduced *from it* the perfectly correct equations (Nos. 4, 5, 6, and 7) nevertheless *always* ceases to exist when any value in cyphers is substituted for x, because any value of x which would make one side of the equation a convergent series, would always make the other a divergent series. The same is also true for the equations (Nos. 4 and 5); they do not exist for any one value in cyphers of x. But nevertheless, the generally correct equation (No. 6) which is deduced from (No. 4), does exist again for every value in cyphers of x, which is >1, and ceases to exist when $x<1$. Similarly for the equation (No. 7) which has been deduced from the equation (No. 5) that does not exist for any single value in cyphers of x, and which is as generally correct as all the others, and which also exists for every value in cyphers of x that is <1, but ceases to exist whenever $x>1$.

Let us now take, in the second place, an example in which a root occurs. Suppose, for example, we have to extract the m^{th} root of the series

$$1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\frac{1}{8!}x^8-\ldots*$$

which occurs in analytical trigonometry under the name of $\cos x$, meaning that we have to find a series proceeding according to whole powers of x, which is equal to the m^{th} root of the above series; we then find in all cases a series R proceeding according to *even* powers of x, so that

$$R^m = \cos x,$$

where $\cos x$ represents the *general* series

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

Now this series R will be convergent at *most* for all values of x between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, because for the next (absolutely) greater values of x, the value of $\cos x$ is negative, and therefore if m is supposed to be *even*, every value of $\sqrt[m]{\cos x}$ will be *imaginary*, while the series R only contains *actual* forms. Hence the series R will be certainly divergent for all values of x which are (absolutely) $> \frac{1}{2}\pi$, and will therefore have no value, and thus shew that it is no longer adapted to represent the value of $\sqrt[m]{\cos x}$ for these values of x. We shall consequently be unable to "calculate" with this *numerical* and *divergent* series (which has resulted for an absolute value of x that is $> \frac{1}{2}\pi$), while the *general* calculation, where we have not yet to regard the value of x, must, according to all that has been said before, necessarily give correct results.

We have a still more simple example in the transformation of $\sqrt{1-x^2}$ into an infinite series. We obtain

$$\sqrt{1-x^2} = \pm\left(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^8-\ldots\right)$$

If we put $x<1$ each of the two series on the right, according as the (+) or (−) sign be taken, is convergent, and its value is equal to one value of $\sqrt{1-x^2}$. But if $x>1$ then each of the series on the right is divergent, and consequently no longer admissible in calculation; it has now no value, and that which has no existence can in no case be equal to the now imaginary value of $\sqrt{1-x^2}$.

But *generally*, where we have not yet regarded any determinate value of x, and when x is therefore a mere supporter of the symbols of operation, the infinite series

$$\pm\left(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^8-\ldots\right)$$

is always *equal* to the root $\sqrt{1-x^2}$, in so far as the root $\sqrt{1-x^2}$ is (by sect. 41) a mere form which only represents the property that when it is potentiated by 2, the result is always $1-x^2$; and since the series itself, whether we retain the (+) or (−) before

* By $n!$ we mean the product $1.2.3\ldots n$, and more generally the factorial $1^{n|1}$ which for $n=1$, and also for $n=0$, is $=1$.

it, has this property, it is *one of the expressions* which is represented by the root $\sqrt{1-x^2}$, and at the same time such an expression, as, on account of its generality, can still be "calculated" with, i. e. as is still admissible in calculation.

And now, in order to give an example of the manner in which analysts now and then neglect the rules which are here given, we shall take, as being nearest at hand, that which will be found in *La Croix's* Treatise upon the Differential and Integral Calculus, (*Traité du Calcul Différentiel et du Calcul Integral.* 2 Ed. 4to. Vol. III. p. 621, seqq.) The problem is there to "sum" the infinite series

$$\text{(R)} \quad \ldots z^r + z_1 . (z-2)^r + z_2 . (z-4)^r + z_3 . (z-6)^r + \ldots$$

upon the hypothesis that z_1, z_2, z_3, &c. represent the consecutive binomial coefficients, and that r is any positive whole number.

Now if the problem had to be solved according to the above rules, we should have to begin by enquiring whether z were to be considered as perfectly general, as a mere supporter of the symbols of operation, whose signification was not to be regarded, or whether z were to represent a determinate value in cyphers, although the same might be still unknown.

In the *first* case, we should have to commence again by transforming the series into another, which proceeded either according to whole powers of z itself, or at any rate of some expression compounded of z, *the coefficients of which series would therefore no longer contain* z *itself.*

But we may soon convince ourselves that if we tried to exhibit such a series, its coefficients, since r is to be a positive whole number, would increase with the number of the terms and would finally become infinitely great (the last, if we may so speak, would be a diverging infinite series), and that consequently the series R does not exist at all in the above mentioned general form.

In the *second* case, where we suppose z to have a determinate actual value in cyphers, although it may be still perfectly unknown, and thus left undetermined, we may as easily convince ourselves that the given series R, considered as numerical, is always divergent.

The series R therefore does not exist if it is considered as *general* in respect to z (r being positive whole); and considered as a *numerical* series it is divergent, i. e. it has no value. It is therefore perfectly impossible to "sum" it, whether we take this word in the more rigorous signification of (sect. 47) or whether we understand by it the determination of the "value" of a numerical (and convergent) series. The problem is consequently impossible in any case.

Nevertheless *Déflers* endeavours to solve this problem by affixing the powers of t to the several terms of the series R, and summing the series.

$$(T_r) \ldots z^r + z_1 . (z-2)^r . t + z_2 . (z-4)^r . t^2 + z_3 . (z-6)^r . t^3 + z_4 . (z-8)^r . t^4 + \ldots$$

for $r = 0, 1, 2, 3$, &c. &c. Let S_r denote the expression which he has found for the sum of the series T_r, then we have the equation

$$T_r = S_r.$$

This expression is now unconditionally correct, generally, i. e. as long as t remains indeterminate, and we can always substitute S_r for T_r, and conversely, *without having to fear any contradictions at any time.* But when we substitute a determinate value in cyphers for t, as e. g. if we put $t=1$, whereby the series T_r will become the series R above given, we must *previously* to deducing the "*value*" or the "*sum*" of this series R from the "sum" S_r of the series T_r, investigate whether the series R really has a value, i. e. whether, when conceived as a numerical series, it is convergent, or whether it can be considered as a general series. Now since, as we have shortly before shewn, neither one nor the other is the case, we can neither talk of its "value" nor of its "sum*." Hence we cannot pretend to find the sum of the series R by putting $t=1$ in S_r, as *Déflers* has done.

Herein we see the explanation of the fact that the conclusions which *Déflers* wished to draw from this summation (loc. cit.) were necessarily false.

But if we suppose that z (in the series R) represents a positive whole number, then the series R terminates, i. e. it is no longer an *in*finite series, and now we of course obtain the value of this finite expression R from the above mentioned sum S_r by putting $t=1$, as is self-evident. In this manner we find that $R=0$ whenever r is an odd number, and that for $r=2$, for example, the value of R is $=z . 2^z$.

SECTION 48[b].

If we now collect all that has been hitherto said about infinite series, we shall find the following practical rules for calculating with them:

(1) We may calculate unconditionally with an infinite series according to the laws of "algebraical sums," or of "whole functions of x," as long as it still proceeds according to the powers of an expression x (which may be simple, or even anywise compounded, but) which is conceived so generally that it is only considered as a mere supporter of the operations (as a constituent element of the *form* with which we are calculating) without regarding its signification in the slightest degree.

(2) But when the series, which occur in such general calcu-

* For we must, as long as we calculate with series, take care that every series with which we calculate can still be conceived as *general*, or, in case it cannot be considered as a general series proceeding according to powers of any expression, that it is convergent. Hence we can no longer calculate with the series R.

lations, become, for particular numerical values of the letters, other series, which can no longer be considered as such as proceed according to the powers of a *general x*, *then we can only allow the results of the general calculation to hold good in the particular case when the series, considered as numerical, are convergent;* because (numerical and) *divergent* series have been acknowledged to be forms, which (as previously the form $\frac{b}{0}$) are inadmissible in calculation, *with which we are therefore no longer permitted to calculate.*

If we follow these rules in calculations with infinite series with proper care, we shall be enabled to calculate with general infinite series as well as with finite expressions, and at the same time be convinced that we can never, either in general, or (what is most important) in any particular case of application, be led into contradictions.

Concluding Remark.

WE can also investigate general infinite series *per se*, which without being equal to finite expressions, and therefore without having a sum in the sense of (sect. 47), yet represent certain properties, i. e. are the supporters of certain properties, *and therefore form ideas* per se. We apply for this purpose the "method of indeterminate coefficients" which has been already employed in (sects. 45, 46); i. e. we assume the form of the series to be

$$A_0 + A_1 . x + A_2 . x^2 + A_3 . x^3 + A_4 . x^4 + \ldots$$

and endeavour to determine the coefficients A_0, A_1, A_2, A_3, A_4, &c. which have been as yet left undetermined, so as to answer the required end.

The commencement of the next chapter will present us with such a problem, while the course of the same chapter will allow us to see, at least in one (but that, a very grand) example, how we can in this way arrive at perfectly general ideas, which include former particular ideas as particular cases. But we shall employ this method "of arriving at new and general *ideas*" in the following pages, only to arrive at a general idea of a power, or rather to arrive at the idea of a general power, because that example is sufficient to shew the general process, and because we are here especially concerned about the idea of the general power, in order to be enabled to conclude our views upon the first part of Mathematical Analysis.

SIXTH CHAPTER.

SECTION 49.

LET us seek an infinite series, which is denoted by f_x, and which has the property that, (as in the elements $a^x . a^z = a^{x+z}$, so)

$f_x . f_z = f_{x+z}$ upon the supposition that f_x, f_z, f_{x+z} represent series which have the *same coefficients*, and which only differ from one another in that one proceeds according to powers of x, another according to powers of z, and the third according to powers of $x+z$.

By means of the method of indeterminate coefficients we find at once that

$$f_x = 1 + cx + \frac{c^2}{2\,!} . x^2 + \frac{c^3}{3\,!} . x^3 + \frac{c^4}{4\,!} . x^4 + \dots$$

in which series c remains perfectly indeterminate and therefore general.

Now let us consider this series in the simplest case where $c = 1$, and let us denote it in this case by ϕ_x, so that

(1) $$\phi_x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \dots$$

then, (since ϕ_x is a particular case of f_x) we have also

(2) $$\phi_x . \phi_z = \phi_{x+z},$$

and, if we put $x - z$ for x and divide by ϕ_z,

(3) $$\frac{\phi_x}{\phi_z} = \phi_{x-z}.$$

Whence it follows further that if m represent any positive or negative whole number or zero

(4) $$(\phi_x)^m = \phi_{mx}\,^*.$$

This equation (No. 4) still holds, when m is positive or negative and broken, provided only that we suppose the dignand to be *positive*, (since we are not yet acquainted with any other broken powers,) as the proof in the note shews†.

If we put $x = 1$ in this equation (No. 4), and x, conceived

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* From (No. 2), viz. it follows that

$$\phi_x . \phi_x = \phi_{2x}; \quad \phi_{2x} . \phi_x = \phi_{3x}; \quad \phi_{3x} . \phi_x = \phi_{4x};$$

hence when m is positive whole $(\phi_x)^m = \phi_{mx}$. And if m is negative whole and for Ex. $= -n$, then we have again $(\phi_x)^m = (\phi_x)^{-n} = \frac{1}{(\phi_x)^n} = \frac{1}{\phi_{nx}} = \frac{\phi_0}{\phi_{nx}}$ (by No. 3) $= \phi_{0-nx} = \phi_{(-n)x} = \phi_{mx}$.

† Put $x : \nu$ for x in (No. 4), and the positive whole number ν for m, then

$$(\phi x : \nu)^\nu = \phi_x, \text{ and therefore } \phi x : \nu = \sqrt[\nu]{\phi_x},$$

where ϕ_x is considered as positive, and the root is the single-meaning positive root;—potentiate this equation by the positive or negative whole number μ, then we find, since by (No. 4)

$$(\phi x : \nu)^\mu = \phi \mu x : \nu,$$

immediately that

$$\phi_{\mu x : \nu} = (\sqrt[\nu]{\phi_x})^\mu = (\phi_x)^{\frac{\mu}{\nu}},$$

provided only that ϕ_x is positive, in order that the root may be the single-meaning positive root, and the power $(\phi_x)^{\frac{\mu}{\nu}}$ the single-meaning *actual* power which was defined and treated in the elements (sects. 26 and 28).

as *actual*, for m, then, (since ϕ_1 is clearly positive) we obtain the correct equation,

(5) $$(\phi_1)^x = \phi_x,$$

that is $\left(1+\frac{1}{1}+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\right)^x = 1+\frac{x}{1}+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots,$

provided x is actual.

If we now once for all denote the positive number ϕ_1 by e, this equation may be written thus

(6) $$e^x = 1+\frac{x}{1}+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots,$$

provided that x is actual, so that e^x has already received a meaning in the elements (sect. 28). And in this equation we have

$$(\odot)\ldots e = 1+\frac{1}{1}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\ldots = 2{,}71825\ldots$$

This equation (No. 6) gives us now an opportunity of extending the idea of the power e^x for imaginary values of x, *by understanding the symbol e^x to denote from henceforth the infinite series* $1+\frac{x}{1}+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots$ *in which x is a mere supporter of the symbols of operation, and may therefore just as well be actual as imaginary.* For e^x now no longer differs from ϕ_x, and the equations (Nos. 2, 3, and 4) may now be written thus

(I.) $$e^x \cdot e^z = e^{x+z},$$

(II.) $$\frac{e^x}{e^z} = e^{x-z},$$

and (III.) $$(e^x)^m = e^{mx},$$

where x and z may be just as well actual as imaginary, while the last equation holds for any positive or negative *whole* m, and *general* x, or for any *actual* m but *positive* e^x. This power e^x is called the *natural* power, and it may be easily proved that it is always convergent for every actual and every imaginary value of x of the form $p+q\,.\,i$, and that it has therefore always an actual or imaginary value of the form $p+q\,.\,i$, and that it is at the same time single-meaning.

Moreover it also follows that

(IV.) $$\sqrt[\nu]{(e^x)} = \sqrt[\nu]{1}\,.\,e^{x:\nu}$$

since by (No. III.)

$$(e^{x:\nu})^\nu = e^{(x:\nu)\cdot\nu} = e^x,$$

provided that ν be supposed positive whole, and $\sqrt[\nu]{1}$ represents any of its ν different values, while x is conceived perfectly general.

The formulæ (Nos. I.—IV.) *are therefore those which can and may be applied for* NATURAL POWERS, {under the restrictions assigned for (Nos. III. and IV.)}.

SECTION 50.

The *natural logarithm* follows instantly upon the natural power, if we mean by it the symbol $\log a$ which denotes any expression x that makes $e^x = a$, that is, $1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = a$. This equation from which x has to be found has the form of the higher algebraical equations, but of an infinite degree, and this circumstance leads us to suppose that $\log a$ (that is x) will have an infinite number of values, which however must all be of the form $p + q \cdot i$.

If we propose to ourselves the problem: to find the values of $\log a$ upon the supposition that a is actual or imaginary but of the form $p + q \cdot i$, that is, if we desire to find all the values of $\log (p + q \cdot i)$, we may represent them by $\alpha + \beta \cdot i$, where α and β are conceived as actual and are most probably infinitely multiple-meaning, but are in any case yet to be found. We then have the equation

$$e^{\alpha + \beta \cdot i} = p + q \cdot i,$$

$$\text{or} \quad e^{\alpha} \cdot e^{\beta \cdot i} = p + q \cdot i.$$

But because $e^{\beta i}$ is found from e^x by putting βi for x, it follows that if we set apart all the terms with even powers of β, and represent the series

$$(1) \qquad 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} + \ldots \text{ by K},$$

and also collect all the terms with uneven powers of β and represent this second series,

$$(2) \qquad \beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \frac{\beta^7}{7!} + \ldots \text{ by } S_\beta,$$

then we shall have

$$(3) \qquad e^{\beta i} = K_\beta + i \cdot S_\beta,$$

$$\text{and } (4) \qquad e^{-\beta i} = K_\beta - i \cdot S_\beta\text{*}.$$

The above equation then separates into

$$\text{(A) } e^{\alpha} \cdot K_\beta = p, \text{ and (B) } e^{\alpha} \cdot S_\beta = q;$$

and it now only remains to find the unknown values of α and β, which are conceived as actual, from these two equations.

For this purpose a more intimate acquaintance with the two series represented by K_β and S_β is required, and the result of this more intimate acquaintance is called *analytical trigonometry*. This more intimate acquaintance must be obtained

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* For *those* readers for whom these pages were put together the remark is needless, that K_β and S_β are the *general* $\cos \beta$ and $\sin \beta$, which here appear, as of themselves, on occasion of the natural power, and *independently of geometry*. Now we must never assume geometry in analysis, since mathematical analysis must according to these views *precede* any doctrine of magnitudes.

before we can think of continuing the solution of the proposed problem.

SECTION 51.

First of all we easily prove that the series K_β and S_β are convergent for all actual and imaginary values of β (of the form $p+q.i$), and have therefore *always* a value, and that they are also only single-meaning.

By solving the equations (sect. 50, Nos. 3 and 4) algebraically with respect to K_β and S_β we obtain

$$K_\beta = \frac{e^{\beta i} + e^{-\beta i}}{2} \text{ and } S_\beta = \frac{e^{\beta i} - e^{-\beta i}}{2 . i},$$

so that these series are exhibited as expressions in powers (commonly called exponential expressions) with which we can more easily "calculate."

Now if we take any equation between powers, as e. g.

$$e^{(x \pm z)i} = e^{xi} . e^{\pm zi},$$

and substitute for the powers the expressions compounded of K and S (by sect. 50, Nos. 3 and 4) we immediately obtain equations between these series K and S, namely the equations,

$$\text{(I.)} \quad S_{x+z} = S_x . K_z + K_x . S_z$$
$$\text{(II.)} \quad K_{x+z} = K_x . K_z - S_x . S_z.$$
$$\text{(III.)} \quad S_{x-z} = S_x . K_z - K_x . S_z.$$
$$\text{(IV.)} \quad K_{x-z} = K_x . K_z + S_x . S_z.$$

And if we multiply the equations (sect. 50, Nos. 3 and 4) together, we find

$$\text{(V.)} \quad 1 = (K_\beta)^2 + (S_\beta)^2.$$

Then it also follows from (Nos. I. and II. for $z = x$)

$$\text{(VI.)} \quad S_{2x} = 2 S_x . K_x.$$
$$\text{(VII.)} \quad K_{2x} = (K_x)^2 - (S_x)^2 = 1 - 2 (S_x)^2 = 2 (K_x)^2 - 1,$$

and from (No. VII. for $x = \frac{1}{2} z$)

$$\text{(VIII.)} \quad S_{\frac{1}{2}z} = \sqrt{\frac{1 - K_z}{2}}.$$

$$\text{(IX.)} \quad K_{\frac{1}{2}z} = \sqrt{\frac{1 + K_z}{2}}.$$

By means of the formulæ (Nos. I.—IV.) the three *products* $S_x . K_z$, $K_x . K_z$, and $S_x . S_z$ may be transformed into *sums and differences*, and also conversely the four *sums* or *differences* $S_\alpha \pm S_\beta$ and $K_\alpha \pm K_\beta$ may be again transformed into *products*.

SECTION 52.

Upon now entering upon the values in cyphers of these general series denoted by K and S, we must first of all remark, that for actual values of x the values of K_x and S_x will be always actual, and (on account of No. V.) must always lie between $+1$ and -1.

Moreover, if we write h for z and substitute for S_h and K_h, the infinite series which are represented by these symbols, the formulæ (Nos. I. and II.) become

$$\text{(X.)}\quad S_{x+h}=S_x+K_x.h-S_x.\frac{h^2}{2!}-K_x.\frac{h^3}{3!}+S_x.\frac{h^4}{4!}+\ldots \text{ in inf.}$$

$$\text{(XI.)}\quad K_{x+h}=K_x-S_x.h-K_x.\frac{h^2}{2!}+S_x.\frac{h^3}{3!}+K_x.\frac{h^4}{4!}+\ldots \text{ in inf.}$$

whence it follows, that:

(*a*) The actual values of the series S_x and K_x alter continuously with the actual values of x.

(*b*) The actual values of the series S_x increase together with those of x, as long as K_x is positive; but decrease continuously, while x is conceived as increasing continuously from $-\infty$ through 0 to $+\infty$, as soon and as long as K_x is negative; they pass moreover from increasing to decreasing (that is, have a *maximum*) at the moment that K_x becomes $=0$, and S_x is positive; and finally pass from decreasing to increasing (i.e. have a *minimum*) at the moment that K_x becomes $=0$, and S_x is negative.

(*c*) The actual values of the series K_x on the other hand decrease, as those of x increase, as long as S_x is positive; increase with the values of x as soon and as long as S_x is negative; pass from $\left\{\begin{matrix}\text{increasing}\\ \text{diminishing}\end{matrix}\right\}$ to $\left\{\begin{matrix}\text{diminishing}\\ \text{increasing}\end{matrix}\right\}$ in the moment that S_x becomes $=0$ and K_x is at the same time $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$.

SECTION 53.

If we suppose the value of S_x (or of K_x) given, and the value of x required, the equation for determining x has always the form of an higher algebraical equation, but of an infinite degree, and this leads us to conjecture that there will be an infinite number of values of x of the form $p+qi$ for which S_x (or K_x) will have one and the same value.

All this points at a periodical recurrence in the values of K_x and S_x. But we are confirmed in this view by a consideration of the equations (sect. 51, Nos. I. and II.) For they shew us that the values of S_{x+z} and S_z, as also of K_{x+z} and K_z are equal to one another whenever the difference x between the arguments (arcs) $x+z$ and z is such that $K_x=1$ and $S_x=0$. The whole investigation now turns upon the discovery of these values.

SECTION 54.

Now if we put $\beta=0$ in (sect. 50, Nos. 1 and 2) we obtain,

(1) $K_0=1$, and (2) $S_0=0$.

Hence, while x increases continuously from 0, S_x will also increase continuously with it from 0 (by sect. 52, *b*), and K_x simultaneously decrease continuously from 1 (by sect. 52, *c*). If we now denote the *smallest* positive value of x, for which K_x has diminished down to 0, (and S_x has therefore simultaneously increased up to 1,) and whose existence may be easily proved*, by $\frac{1}{2}\pi$, that is, the double of this positive number by π, then we have,

(3) $K_{\frac{1}{2}\pi} = 0$ and (4) $S_{\frac{1}{2}\pi} = 1$.

And we now deduce by means of the equations (sect. 52, Nos. VI. VII.) for $x = z = \frac{1}{2}\pi$ or $x = z = \pi$, and (sect. 52, Nos. I. and II.) for $x = \pi$, $z = \frac{1}{2}\pi$, the following results

(5) $K_\pi = -1$ and (6) $S_\pi = 0$,

(7) $K_{\frac{3}{2}\pi} = 0$ and (8) $S_{\frac{3}{2}\pi} = -1$,

(9) $K_{2\pi} = +1$ and (10) $S_{2\pi} = 0$.

Then it follows further from (sect. 52, Nos. I.—IV.) for $x = \pi$, 2π,

(11) $K_{\pi - z} = -K_z$ (12) $S_{\pi - z} = +S_z$,

(13) $K_{\pi + z} = -K_z$ (14) $S_{\pi + z} = -S_z$,

(15) $K_{2\pi - z} = +K_z$ (16) $S_{2\pi - z} = -S_z$.

If we now separate all *positive* (whole or broken, rational or irrational) contiguous numbers into equal sections (from 0 to $\frac{1}{2}\pi$, from $\frac{1}{2}\pi$ to π, from π to $\frac{3}{2}\pi$, from $\frac{3}{2}\pi$ to 2π, from 2π to $\frac{5}{2}\pi$ and so on), the limits of any one of which always exceed those of the previous one by $\frac{1}{2}\pi$, and if we call the collection of all positive numbers in any such section a *quadrant*, we shall see clearly from the formulæ (Nos. 1—16.) (by supposing all numbers in the first quadrant, i. e. from 0 to $\frac{1}{2}\pi$ to be substituted for z), that:

Within the four first quadrants, while the actual values of x are supposed to increase continuously from 0 to 2π, the values of the series K_x and S_x are as follows:

Quadrants.	Values of the series K_x.	Values of the series S_x.
in the 1st.	positive, and decreasing from 1 to 0.	positive and increasing from 0 to 1.
in the 2nd.	negative and decreasing from 0 to − 1.	positive and decreasing from 1 to 0.
in the 3rd.	negative and increasing from − 1 to 0.	negative and decreasing from 0 to − 1.
in the 4th.	positive and increasing from 0 to 1.	negative and increasing from − 1 to 0.

And then we easily find from (sect. 52, Nos. I. and II.) (for $x = 2\pi$, 4π, 6π, ... and $z = 2\pi$), that when n is a positive whole number,

(17) $K_{2n\pi} = 1$ and (18) $S_{2n\pi} = 0$,

* We shew that for $x = 1$, K_x is still positive, but that for $x = 2$, K_x has become negative, consequently there is one value of x between 1 and 2, for which $K_x = 0$. Hence not only does this value exist, but it may even be found by the Newtonian method of approximation.

and further (sect. 58)

(19) $K_{8n\pi+y} = K_y$ and (20) $S_{8n\pi+y} = S_y$

that is, in any four following quadrants the values of the series K_x and S_x recur exactly in the same order as in the four first quadrants.

Finally it follows from (sect. 50, Nos. 1 and 2) that

(21) $K_{-x} = K_x$ and (22) $S_{-x} = -S_x$,

whence it follows that the formulæ (Nos. 17—20) hold whether n be positive or negative whole, or zero.

SECTION 55.

It now clearly follows, that

(a) The values of the series K_x and S_x may be calculated for all positive and negative values of x by the formulæ (Nos. 11—22) without any difficulty as soon as they have been all calculated and tabulated for all positive values z of x in the first quadrant;

(b) The values of the series K_x and S_x may be also calculated for all imaginary values of x which $= p + qi$, as soon as the values of $K_{\beta i}$ and $S_{\beta i}$, that is, (by sect. 50, Nos. 1 and 2) the values of the series

$$1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \frac{\beta^6}{6!} + \ldots \text{ or } \frac{e^{\beta} + e^{-\beta}}{2},$$

$$\text{and } \beta + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \frac{\beta^7}{7!} + \ldots \text{ or } \frac{e^{\beta} - e^{-\beta}}{2},$$

have been calculated and tabulated for *all* positive values of β*, since by (Nos. I. and II.) of (sect. 52)

$$S_{p+q.i} = S_p \cdot K_{qi} + K_p \cdot S_{qi},$$

$$\text{and } K_{p+q.i} = K_p \cdot K_{qi} - S_p \cdot S_{qi}.$$

SECTION 56.

Hence if we have given $K_x = \mu$ and $S_x = \nu$, where μ and ν may be any actual or imaginary expressions and we find that $\mu^2 + \nu^2 = 1$, then x has an infinite number of values which satisfy both these equations, and if ϕ is any one of these values actual or imaginary, then *all* these values are represented by $\pm 2n\pi + \phi$, when n represents zero or any positive number. If μ and ν are actual, then one of these values of x (and only one) lies within the four first quadrants, and this may be that represented by ϕ. It may be also easily proved that besides the infinitely many values of x represented by $\pm 2n\pi + \phi$, there is no other one which can simultaneously satisfy both equations $K_x = \mu$ and $S_x = \nu$.

If K_x or S_x is given by itself, then x has twice as many

* We possess the commencement of such a table calculated by *Gudermann*.

values as if K_x and S_x were given simultaneously, because we may just as well take $S_x = +\sqrt{1-\mu^2}$ as $S_x = -\sqrt{1-\mu^2}$ for $K_x = \mu$, and conversely.

SECTION 56ᵇ.

(Common) calculation with cyphers may now be somewhat further extended. The object of (common) calculation with cyphers is, namely: "to *transform* any expression, however simple or however complex, which has originally arisen from numerical numbers (sect. 31), into a *general numerical number* (sect. 39), i. e. into an *actual* number (positive or negative whole or broken number, or zero) or into an *imaginary* number of the form $p+q.\sqrt{-1}$, where p and q are actual, and secondarily, if required, to express the actual whole numbers, or the numerators and denominators of the actual broken numbers in numerical numbers (sect. 31), i. e. in sums arranged according to powers of 10, unless it is preferred to exhibit the fractions as decimal fractions." And an expression is said to be "*calculated**" or "*evaluated*" when it has been reduced to this last (actual or imaginary) form $p+q\sqrt{-1}$.

Now since *all* actual values are also contained in the form $p+q\sqrt{-1}$ (in as much as q may be $=0$), we shall be able to accomplish the object of (common) calculation with cyphers thoroughly, if we learn how to transform (1) the *sum*, (2) the *difference*, (3) the *product*, and (4) the *quotient* of two such *general numerical* expressions as $\alpha+\beta\sqrt{-1}$ and $\gamma+\delta\sqrt{-1}$, into a similar expression of the form $p+q.\sqrt{-1}$; and also the expressions (5) $\sqrt[m]{p+q\sqrt{-1}}$, (6) $(p+q\sqrt{-1})^{\frac{\mu}{\nu}}$, (7) $\log(p+q\sqrt{-1})$, and (8) $(p+q\sqrt{-1})^{\alpha+\beta\sqrt{-1}}$ into a similar expression of the same form $P+Q\sqrt{-1}$. The four first of these problems have been already perfectly solved (in sect. 39), while we have as yet only treated the particular case of the fifth problem (in sect. 39) where, instead of the m^{th} we had the *square* root. The four last problems of (common) calculation with cyphers must consequently be solved in this place.

We shall proceed to solve the 5th and 6th of these problems, as a preliminary step, in the next (sect.), and then in the following (sect. 58) come to the solution of the 7th problem, which was proposed in (sect. 50), and finally, after we have at last been able to establish the idea of the *general* power, we shall solve the last of the above problems, and thus be enabled to bring (common) calculation with cyphers to a conclusion.

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* "To calculate an expression" (einen Ausdruck *berechnen*) must be carefully distinguished from "to calculate *with* an expression" (*mit* einem Ausdrucke rechnen) that is, (sect. 56ᵇ) must not be confounded with (sect. 6). *Trans.*

SECTION 57.

For this purpose we prove the formulæ, when μ is positive or negative whole, while ν is only positive;

$$\text{(I.)}\quad (K_\beta + i . S_\beta)^\mu = K_{\mu\beta} + i . S_{\mu\beta};$$

$$\text{(II.)}\quad \sqrt[\nu]{K_\beta + i . S_\beta} = K_{(2n\pi+\beta):\nu} + i . S_{(2n\pi+\beta):\nu};^*$$

where zero or any positive or negative whole number may be substituted for n, while the expression on the right in (No. II.) will nevertheless furnish no more than ν values of the root on the left, since the values of $K_{(2n\pi+\beta):\nu}$ and $S_{(2n\pi+\beta):\nu}$ are not altered when n is increased by a multiple of ν (by sect. 54, Nos. 19 and 20). Consequently we only give the ν values 0, 1, 2, 3, ... $\nu-1$, to n on the right in (No. II.), or often only half these positive values, with the addition of the negative values -1, -2, -3, &c. until we have really obtained the ν different values of the root on the left.

If we now write $\mu\beta$ for β in (No. II.) we obtain also

$$\text{(III.)}\quad \sqrt[\nu]{(K_\beta + i . S_\beta)^\mu} = K_{(2n\pi+\mu\beta):\nu} + i . S_{(2n\pi+\mu\beta):\nu},$$

while on the other hand by potentiating (No. II.) on the right and left by μ, we find from (No. II.)

$$\text{(IV.)}\quad \{\sqrt[\nu]{K_\beta + i . S_\beta}\}^\mu = K_{(2n\pi+\beta)\mu:\nu} + i . S_{(2n\pi+\beta)\mu:\nu};$$

where we must in all cases substitute for n *at most* the ν values 0, 1, 2, 3 ... $\nu-1$, or only half these positive values, and then the negative ones -1, -2, -3, &c. Now it may be seen from the formulæ (Nos. III. and IV.), that there will be always ν really *different* values on the right in (No. III.), but that the number of the different values in (No. IV.) is less, whenever μ and ν have a common divisor. And herein we have for the first time the more general explanation of the assertion made in (sect. 41 upon No. 4.), viz. that $\sqrt[\nu]{(a^\mu)} = (\sqrt[\nu]{a})^\mu$ is not a correct equation in the sense of (sect. 3).

Finally, the expressions on the right in these 4 equations (Nos. I.—IV.) express exactly *all* the values of the expressions on the left, neither more nor less, so that these equations (Nos. I.—IV.) are perfectly correct equations in the sense of (sect. 3).

Now if p and q are any actual numbers, and we have to

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* For by (sect. 50, No. 3.), $K_\beta + i . S_\beta = e^{\beta i}$ and $(e^{\beta i})^\mu = e^{\mu\beta i} = K_{\mu\beta} + i . S_{\mu\beta}$, as also by (sect. 54, Nos. 19 and 20) $K_\beta + i . S_\beta = K_{2n\pi+\beta} + i . S_{2n\pi+\beta} = e^{(2n\pi+\beta)i}$, and therefore $\sqrt[\nu]{(K_\beta + i . S_\beta)} = \sqrt[\nu]{e^{(2n\pi+\beta) . i}} =$ (by sect. 49. IV.) $e^{(2n\pi+\beta)i:\nu} . \sqrt[\nu]{1}$
$= \sqrt[\nu]{1} \times [K_{(2n\pi+\beta):\nu} + i . S_{(2n\pi+\beta):\nu}]$.
But because the second factor on the right has already ν different values, and the whole result on the right cannot have more than ν values, we may put, for $\sqrt[\nu]{1}$ the single value 1. From this results No. II.

potentiate $p+q.i$ by μ, or to "calculate" $\sqrt[\nu]{p+q\,i}$, or the root $\sqrt[\nu]{(p+q\,i)^\mu}$ or the power $\{\sqrt[\nu]{p+q\,i}\}^\mu$, i.e. to reduce the value of these expressions to the form $P+Q.i$, we proceed as follows:

Calculate $r=+\sqrt{p^2+q^2}$, and ϕ as the *least positive* value that satisfies the equations $K_\phi=\frac{p}{r}$ and $S_\phi=\frac{q}{r}$, so that $p+q.i=r.(K_\phi+i.S_\phi)$, then we have by the assistance of Nos. I.—IV.

(1) $(p+q.i)^\mu=r^\mu\times[K_{\mu\phi}+i.S_{\mu\phi}]$;

(2) $\sqrt[\nu]{p+q.i}=\sqrt[\nu]{r}\times[K_{(2n\pi+\phi):\nu}+i.S_{(2n\pi+\phi):\nu}]$;

(3) $\sqrt[\nu]{(p+q.i)^\mu}=\sqrt[\nu]{(r^\mu)}\times[K_{(2n\pi+\mu\phi):\nu}+i.S_{(2n\pi+\mu\phi):\nu}]$;

(4) $(\sqrt[\nu]{p+q.i})^\mu=(\sqrt[\nu]{r})^\mu\times[K_{(2n\pi+\phi)\mu:\nu}+i.S_{(2n\pi+\phi)\mu:\nu}]$,

where $\sqrt[\nu]{r}$ and $\sqrt[\nu]{r^\mu}$ are the single-meaning positive roots and powers which have been already established in the elements (sects. 25, 26, 28), while it is sufficient to give n the ν values 0, 1, 2, 3, 4, ... $\nu-1$, or only the half of these positive values, and as many negative values -1, -2, -3, &c. &c.

SECTION 57[b].

We call *that* one of the ν values of $\sqrt[\nu]{p+p.i}$ in (No. 2), the *simplest*, in which n is taken $=0$, i.e. which does *not* receive the number π. But if $p+q.i$ is actual and also positive (and $=a$ suppose), then the simplest value of $\sqrt[\nu]{a}$ is at the same time the positive or absolute root of (sect. 26).*

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* If we wish to obtain the ν values of $\sqrt[\nu]{1}$ and of $\sqrt[\nu]{-1}$ from (No. 2), we must take $q=0$, and $p=+1$ or $p=-1$. Then $r=+1$, $\sqrt[\nu]{r}=1$, and for $p=+1$, $\phi=0$, for $p=-1$, $\phi=\pi$, and we have

$$\sqrt[\nu]{1}=K_{2n\pi:\nu}+i.S_{2n\pi:\nu}=e^{2n\pi i:\nu},$$
$$\sqrt[\nu]{-1}=K_{(2n+1)\pi:\nu}+i.S_{(2n+1)\pi:\nu}=e^{(2n+1)\pi i:\nu},$$

in which n receives successively the ν values 0, 1, 2, 3 ... $\nu-1$, or only half of these positive values, and instead of the other half, the negative values -1, -2, -3, &c. &c.

We can also obtain these results directly from the formula (No. II.), remembering that $+1=K_{2n\pi}+i.S_{2n\pi}$ and $-1=K_{(2n+1)\pi}+i.S_{(2n+1)\pi}$. And then we have directly

$$\sqrt[\nu]{1}=\sqrt[\nu]{K_{2n\pi}+i.S_{2n\pi}}=\sqrt[\nu]{(e^{2n\pi i})}=e^{(2n\pi:\nu)i}=K_{2n\pi:\nu}+i.S_{2n\pi:\nu};$$
$$\sqrt[\nu]{-1}=\sqrt[\nu]{K_{(2n+1)\pi}+i.S_{(2n+1).\pi}}=\sqrt[\nu]{\{e^{(2n+1)\pi i}\}}=e^{[(2n+1)\pi:\nu]i}$$
$$=K_{(2n+1)\pi:\nu}+i.S_{(2n+1)\pi:\nu}.$$

The *simplest* value of these roots (which according to the above definition results from putting $n=0$) is consequently 1 for $\sqrt[\nu]{1}$, and $K_{\pi:\nu}+i.S_{\pi:\nu}$ for $\sqrt[\nu]{-1}$. But the last value is not the actual value -1, which exists when ν is uneven, and which results from putting $2n+1=\nu$.

SECTION 58.

The problem of (sect. 50) may now be completly solved, that is, the natural logarithm of $p+q.i$ can now be "calculated." We were there led, by putting

$$\log(p+qi)=\alpha+\beta.i, \text{ and therefore } p+qi=e^{\alpha+\beta i}$$

to the equations,

$$\text{(A) } e^{\alpha}.\mathrm{K}_{\beta}=p, \quad \text{and (B) } e^{\alpha}.\mathrm{S}_{\beta}=q.$$

From these we now find (by squaring and adding)

$$\text{(C) } e^{\alpha}=+\sqrt{p^2+q^2}=r,$$

so that α shews itself to be the actual logarithm (sect. 29) of the positive number r for the positive base e, but has never more than one single value, which may be denoted by $\mathrm{L}r$, so that $\mathrm{L}r$ denotes the one single actual value of the natural logarithm of r (which is found to be negative, zero, or positive according as r itself is <1, $=1$ or >1).

Besides this we have from the equations (A and B)

$$\text{(D) } \mathrm{K}_{\beta}=\frac{p}{r} \text{ and } \mathrm{S}_{\beta}=\frac{q}{r}$$

and for these actual values of K_{β} and S_{β} we find all the values of $\beta=\pm 2n\pi+\phi$, where ϕ lies between 0 and 2π, and denotes the *least positive* one of these values. Therefore we have

$$\text{(I.)} \qquad \log(p+qi)=\mathrm{L}r+(2n\pi+\phi).i$$

where n represents every positive, and also every negative whole number.

Remark. We must not overlook the circumstance that the equation (D) presupposes that r is not zero, that is, that we have not simultaneously $p=0$ and $q=0$. The values which we have found for the natural logarithm consequently always cease to be admissible in calculation, whenever the logarithmand $=0$.

Hence $\log 0$ is a form which is as inadmissible in calculation as $\frac{b}{0}$, and when we therefore, in the applications of any general calculation, encounter $\log 0$, the calculation must be immediately given up, and the present particular result particularly investigated.

SECTION 58[b].

Among these infinitely many values of $\log(p+qi)$, which are all really different from one another, i. e. not equal to one another, and which will with few exceptions be always imaginary, we will term that one *the simplest* which results from putting $n=0$, and therefore does not contain the number π. But if $a=p+qi$ is actual and positive, that is, if $p=a$ and $q=0$, and therefore $r=a$ and $\phi=0$, this simplest value of $\log a$ is also the actual one which we have just represented by $\mathrm{L}a$.

Consequently we can in general, also, even when a does not happen exactly to be positive, but may be either negative or imaginary, denote the SIMPLEST *value of* $\log a$ *by* $\mathrm{L}\,a$, so that the signification of $\mathrm{L}\,a$ is thereby generalized; and we have

$$\text{(II.)} \qquad \mathrm{L}\,(p+qi) = \mathrm{L}\,r + \phi\,.\,i,$$

$$\text{and (III.)} \qquad \log a = \mathrm{L}\,a + 2n\pi\,.\,i,$$

where n represents zero, or any positive or negative whole number, while a may be either actual or imaginary.

If on the other hand a is actual and positive, we have also

(1) $\log a = \mathrm{L}\,a + 2n\pi\,.\,i,$

(2) $\log(-a) = \mathrm{L}\,a + (2n+1)\,\pi\,.\,i = \mathrm{L}\,(-a) + 2n\pi\,.\,i,$

(3) $\log 1 = 2n\pi\,.\,i,$

(4) $\log(-1) = (2n+1)\,\pi\,.\,i = \mathrm{L}\,(-1) + 2n\pi\,.\,i$*,

where $\mathrm{L}\,a$ represents the *simplest* value of $\log a$, and is in this case, where a is *positive*, also the single actual value of the natural logarithm of a, i. e. the logarithm which is known in the elements (sect. 29) under the name of the *actual* logarithm (of the positive number a, for the positive base e), whilst, when $-a$ is negative, we shall still have

$$\mathrm{L}\,(-a) = \mathrm{L}\,a + \pi\,.\,i.$$

SECTION 59.

With these (single-meaning) *simplest* values of the natural logarithm, we can only calculate according to the two formulæ,

(1) $\mathrm{L}\,(ab) = \mathrm{L}\,a + \mathrm{L}\,b,$

(2) $\mathrm{L}\left(\dfrac{a}{b}\right) = \mathrm{L}\,a - \mathrm{L}\,b,$

where a and b are any actual or imaginary numbers.

With regard to the remaining formulæ which were previously given (sect. 30) for *actual* logarithms, we cannot as yet say that *generally* $\mathrm{L}\,(a^x) = x\,.\,\mathrm{L}\,a$, because we have not yet established the meaning of a^x when a and x are general. Finally, the equation $\mathrm{L}\,(\sqrt[m]{a}) = \dfrac{\mathrm{L}\,a}{m}$, where m is a positive whole number, cannot be considered as an equation in the sense of (sect. 3), because $\sqrt[m]{a}$, and consequently also $\mathrm{L}\,(\sqrt[m]{a})$, has m values, whereas the expression on the left only represents one single value, so that one side of this equation cannot be unconditionally substituted for the other.

For if we put $a = p + qi$, $r = +\sqrt{p^2+q^2}$, $\frac{p}{r} = \mathrm{K}_\phi$, and $\frac{q}{r} = \mathrm{S}_\phi$, and we understand by ϕ itself the *least positive* value of ϕ, then we have by (sect. 57, No. II.)

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* For by (No. II.) $\mathrm{L}(-a) = \mathrm{L}\,a + \pi\,.\,i,$
and $\mathrm{L}(-1) = \mathrm{L}\,1 + \pi\,.\,i = \pi\,.\,i.$

$$\sqrt[m]{a}=\sqrt[m]{p+q.i}=\sqrt[m]{r}.\{K_{(2n\pi+\phi):m}+i.S_{(2n\pi+\phi):m}\},$$

that is, $\sqrt[m]{a}=\sqrt[m]{r}.e^{[(2n\pi+\phi):m]i}$,

where $\sqrt[m]{r}$ represents the single-meaning absolute root, while n is zero, and all whole numbers up to $m-1$. Therefore

$$(a)\quad L(\sqrt[m]{a})=L(\sqrt[m]{r})+\frac{2n\pi+\phi}{m}.i=\frac{Lr+(2n\pi+\phi)i}{m},$$

because $\sqrt[m]{r}$ is positive, $L(\sqrt[m]{r})$ is actual, and for actual logarithms the equation $L(\sqrt[m]{r})=\frac{Lr}{m}$ has already been shewn to be correct (sect. 30.) On the other hand, since $(p+qi)=r(K_\phi+i.S_\phi)=r.e^{\phi.i}$,

we have $La=L(p+q.i)=Lr+\phi.i$,

and therefore $(b)\quad \frac{La}{m}=\frac{Lr+\phi.i}{m}$.

If we now compare the two results in (a) and (b) on the right, we find that,

$$(3)\qquad L(\sqrt[m]{a})=\frac{La}{m}$$

is a correct equation only when we also put for $\sqrt[m]{a}$ its *simplest* value, i. e. that value in whose imaginary portion the number π does not occur, (see sect. 57[b]). Hence if a is actual and positive, the equation (No. 3) holds, provided the positive value of this root is meant by $\sqrt[m]{a}$, and it then coincides with that of (sect. 30).

SECTION 60.

Finally, with regard to the *infinitely multiple-meaning* natural logarithms, we must calculate with them with the same precaution, as we find inculcated in (sects. 37, and 41) for the many-meaning roots.

The formulæ

$$(1)\quad \log(ab)=\log a+\log b;^*$$

$$(2)\quad \log\left(\frac{a}{b}\right)=\log a-\log b;$$

$$(3)\quad \log(\sqrt[m]{a})=\frac{\log a}{m},$$

have on the right and the left the same number of and precisely the same values, provided $\sqrt[m]{a}$ be considered as having m mean-

* Hence $\log(a^2)=\log a+\log a$; on the other hand we may not write $2\log a$ for $\log a+\log a$ (comp. sect. 37 and sect. 41), because $2\log a$ has fewer values than $\log a+\log a$, viz. it is *without* all those values of the latter sum ($\log a+\log a$) for which the summands are conceived as representing different values. Namely, we find by a more accurate investigation that $\log(a^2)$ possesses all the values of $2\log a$ and of $2\log(-a)$. Consequently when *Bernouilli* concluded from $\log(a^2)=2\log a$, and $\log\{(-a)^2\}=2\log(-a)$, that $\log a=\log(-a)$, he made precisely the same mistake, as if we were to conclude from the equations

$$\sqrt{a^2}=+a, \text{ and } \sqrt{a^2}=-a, \text{ that } +a=-a.$$

ings;—these equations are therefore (by sect. 3) unconditionally correct; i.e. the two expressions to the right and left of these equations can be unconditionally substituted one for the other.

On the other hand we cannot yet say that in general $\log(a^x) = x \log a$, because a^x has yet no general meaning; and for the particular case that a is general while x is a positive or negative whole number, $\log(a^x)$ has always many ($\pm x$ times) more values than $x \log a$, as the following investigation will shew.

For if $a = p + q.i$, and r and ϕ have their former meanings, so that ϕ lies within the four first quadrants, we have

$(\alpha)\quad \log(a^x) = \log\{(p+q.i)^x\} = \log(r^x.e^{x\phi.i}) = x\mathrm{L}r + x\phi i + 2n\pi.i,$

where n represents zero and any positive and negative whole number. While

$(\beta)\quad x\log a = x\log(p+q.i) = x\{\mathrm{L}r + (2n\pi+\phi)i\} = x\mathrm{L}r + x\phi i + 2nx\pi.i.$

If we compare (α) and (β) on the right we find that in (α) there are $\pm x$ times as many values as in (β). And the assertion in the note (for $x=2$) is at the same time justified.

Section 61.

Now since all the infinite series f_x, which were sought and found in (sect. 49), and have the property that $f_x.f_y = f_{x+y}$, may be expressed by e^{cx}, the *general* power a^x, which we are endeavouring to find, must be contained in e^{cx}. Now since for $x = 1$, a^x becomes a, and for $x = 1$, e^{cx} becomes e^c, we must take $e^c = a$, and therefore either $c = \log a$ or $c = \mathrm{L}a$, where $\log a$ has infinitely many values, and $\mathrm{L}a$ represents the simplest of these values.

We can therefore establish the idea of the power a^x (in which a and x are any actual or imaginary numbers, so that we may always suppose

$$a = p + q.i \text{ and } x = \alpha + \beta.i)$$

perfectly generally as a *form* a^x which represents the natural power $e^{x\log a}$. This power a^x we therefore call the *general power*.

Now $\log a$, that is, $\log(p+qi)$ has infinitely many values, which (if we put $r = +\sqrt{p^2+q^2}$, and if ϕ represents the *least* positive value which results from the equations

$$\mathrm{K}_\phi = \frac{p}{r}, \text{ and } \mathrm{S}_\phi = \frac{q}{r},$$

and $\mathrm{L}r$ the actual logarithm of the positive number r for the positive base e), are all represented by

$$\log a = \mathrm{L}r + (2n\pi + \phi).i,$$

where n represents zero or any positive or negative whole number;—therefore the *general power* a^x or $e^{x.\log a}$ expresses an infinite number of such infinite series (i.e. of such natural powers). We separate from these one value, corresponding to the *simplest* value of $\log a$ (58[b]), which consequently does not receive the number n; and is the value $e^{x.\mathrm{L}a}$; this we still represent by a^x, but call the *simplest value of the general power.*

The symbol a^x therefore represents, at one time *all* the infinitely many values which are expressed by $e^{x.\log a}$, and is then called the *general power;* while at another time the same symbol a^x only represents the *simplest value* $e^{x \cdot La}$ *of this general power* a^x.

In all cases therefore in which any ambiguity arises, or may arise from the double-meaning of the same symbol a^x, we must necessarily meet the difficulty in a decided manner, either by words or by particular distinctive symbols.

SECTION 62.

OF THE SIMPLEST VALUES OF THE GENERAL POWER.

We shall now however first consider the simplest values of the general power. According to the definition we can immediately "calculate" these powers, i.e. reduce them to the form $P + Q . i$; for, on the hypothesis that we are only employing the simplest values of the powers,

($\odot$) a^x that is, $(p+q.i)^{\alpha+\beta.i} = e^{(\alpha+\beta.i).L(p+q.i)} = e^{(\alpha+\beta.i).(Lr+\phi i)}$

$$= e^{\alpha.Lr-\beta\phi}.(K_{\beta.Lr+\alpha\phi} + i.S_{\beta.Lr+\alpha\phi}),$$

where $r = +\sqrt{p^2+q^2}$ and ϕ is the least positive value found for ϕ from the equations $K_\phi = \frac{p}{r}$ and $S_\phi = \frac{q}{r}$, while Lr represents the actual logarithm of r.

Now if $a = p + qi$ is *actual* and *positive*, we obtain from this equation since $q = 0$, $p = a$, $r = a$, $\phi = 0$,

$$(\mathbb{D})\quad (+a)^{\alpha+\beta i} = e^{\alpha.La}.(K_{\beta.La} + i.S_{\beta.La})$$
$$= a^\alpha.(K_{\beta.La} + i.S_{\beta.La}),$$

where a^α is actual; and the Author has in his Instruction-Books, considered this particular case of the simplest value of the general power under the name of the *artificial* power (*künstliche Potenz*) (inasmuch as the artificial logarithm is opposed to it).

Now it is self-evident (from $\odot$ and $\mathbb{D}$) that

(1) If x is actual and $= \frac{\mu}{\nu}$, and a is positive, then $\beta = 0$, $\alpha = \frac{\mu}{\nu}$, $q = 0$, $p = a$, $\phi = 0$, and therefore from ($\mathbb{D}$)

$$a^{\frac{\mu}{\nu}} = e^{\frac{\mu}{\nu}.La} = \sqrt[\nu]{(e^{\mu.La})} = \sqrt[\nu]{a^\mu} \text{ (by No. 10, sect. 49),}$$

that is, the simplest value of the general power is in this particular case, where the dignand is positive, and the exponent x actual, at the same time the *actual* power of (sect. 28).

(2) If x is positive or negative, whole or zero, but a general, and $= p + q.i$, then $\beta = 0$, $\alpha = x$, positive or negative, whole or zero, and (from $\odot$)

$$a^x = (p+q.i)^x = r^x.(K_{x\phi} + i.S_{x\phi}),$$

where (by No. 1) r^x represents the actual, and therefore here the difference-power (of sect. 24). Consequently the simplest value of the general power a^x is in this case at the same time the difference-power of (sect. 24), by (sect. 57, No. 1).

(3) If $a = e$, but $x = \alpha + \beta i$ be anywise actual or imaginary, then $La = Le = 1$, and the simplest value a^x or $e^{x \cdot La}$ of the general power, coincides with the *natural power* e^x.

The simplest value of the general power a^x consequently includes all powers hitherto considered as particular cases; *and the definition given in (sect. 61) is therefore justified.*

For these simplest values of the general powers the five formulæ or laws may be immediately proved (from $a^x = e^{x \cdot La}$, $a^z = e^{z \cdot La}$), viz.

$$(1)\quad a^x . a^z = a^{x+z};\quad (2)\quad a^x : a^z = a^{x-z};$$

$$(3)\quad a^x . b^x = (ab)^x;\quad (4)\quad a^x : b^x = (a:b)^x,$$

$$\text{and } (5)\quad (a^x)^z = a^{xz},$$

so that we can apply them without further trouble for the purpose of "calculating" with the *simplest values* of the general powers, although a, x, z, b are here conceived to be either actual or imaginary.

At the same time it is clear from the idea of the simplest value of the general power, that although

$$(6)\quad L(a^x) = x . La,$$

however general a and x may be conceived as being, actual or imaginary, provided L represent here, as usual, the simplest value of the natural logarithm, yet that the equation

$$\log (a^x) = x \log a,$$

where log represents all the infinitely many values of the natural logarithm, only holds in a very limited signification.

For (α) $\log(a^x) = L(a^x) + 2n\pi . i = x . La + 2n\pi . i,$
while (β) $x \log a = x . La + 2n'x\pi . i.$

Now these two expressions agree in their actual portions, but not in their imaginary parts; not even when x is positive or negative whole, since $x \log a$ has then much fewer values than $\log(a^x)$ so that the two expressions $\log(a^x)$ and $x \log a$ cannot be unconditionally substituted for one another*. Hence, (according to the idea of sect. 3) $\log(a^x)$ and $x \log a$ are in general *not* equal.

Remark. According to the remark in (sect. 58) and from an inspection of the above formula (⊙), the idea of the general power excludes the case, where the dignand $p + qi$ becomes $= 0$, that is, where $p = q = r = 0$. But if we conceive $q = 0$, and p, and therefore r to decrease continually, then Lr although always negative, will increase absolutely. If then α is *positive* and $\beta = 0$, the first factor (in ⊙) will approach zero nearer and nearer, the smaller p becomes, and thus we shall find at last that $0^\alpha = 0$, *provided that* α *is positive*, either rational or irrational.

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* Compare note to (sect. 60). If the above truth be taken in consideration, another source of the paradoxes of calculation will be dried up.

SECTION 68.

A *general logarithm* is opposed to this simplest value of the general power. We understand, viz. by the general logarithm, the symbol $\overset{c}{\log}\, a$, in which a and c are any actual or imaginary numbers, and which denotes any expression x, which makes the simplest value of the general power $c^x = a$, i. e. $e^{x.\mathrm{L}c} = a$.

From this definition it follows that,

$$\text{(I.)} \qquad \overset{c}{\log}\, a = \frac{\log a}{\mathrm{L}c} = \frac{\mathrm{L}a + 2n\pi . i}{\mathrm{L}c},$$

where the numerator $\log a$, denotes all the infinitely many values of the natural logarithm of a, while $\mathrm{L}a$ and $\mathrm{L}c$ denote the *simplest* values of the natural logarithm of the number a and the base c, (so that for example $\mathrm{L}c$ is the *actual* logarithm, whenever the base c is considered as actual and positive). In this last case (when the base is positive and actual) the general logarithm becomes that which is termed the *artificial logarithm* in the Author's Instruction-books.

If then we wish to "calculate" the general logarithm $\overset{p+qi}{\log}(\gamma + \delta i)$, that is, to reduce it to the form $\mathrm{P} + \mathrm{Q} . i$, we have

$$\text{(II.)} \quad \overset{p+qi}{\log}(\gamma + \delta i) = \frac{\log(\gamma + \delta i)}{\mathrm{L}(p + qi)} = \frac{\mathrm{L}\rho + (2n\pi + \psi) . i}{\mathrm{L}r + \phi . i}$$

$$= \frac{\mathrm{L}\rho . \mathrm{L}r + (2n\pi + \psi)\phi}{(\mathrm{L}r)^2 + \phi^2} + \frac{-\phi . \mathrm{L}\rho + (2n\pi + \psi)\mathrm{L}r}{(\mathrm{L}r)^2 + \phi^2} . i,$$

where $r = +\sqrt{p^2 + q^2}$, $\rho = +\sqrt{\gamma^2 + \delta^2}$, and where ϕ and ψ are the least positive values found from the equations,

$$\mathrm{K}_\phi = \frac{p}{r}, \ \mathrm{S}_\phi = \frac{q}{r} \text{ and } \mathrm{K}_\psi = \frac{\gamma}{\rho}, \ \mathrm{S}_\psi = \frac{\delta}{\rho},$$

while n may represent zero or any positive or negative whole number.

Now it follows from hence, that:

(A) The general logarithm has always infinitely many values, which are either all imaginary, or of which only one is actual.

(B) If the base $p + qi$ is positive and $= a$, then $r = a$, $\phi = 0$, and we have,

$$\text{(III.)} \quad \overset{a}{\log}(\gamma + \delta i) = \frac{\mathrm{L}\rho + (2n\pi + \psi) . i}{\mathrm{L}a} = \frac{\mathrm{L}\rho}{\mathrm{L}a} + \frac{2n\pi + \psi}{\mathrm{L}a} . i,$$

and these are therefore the values of the *artificial* logarithm of $\gamma + \delta . i$. They are all imaginary, when $\gamma + \delta . i$ is imaginary or negative, but there is one actual value $\left(\frac{\mathrm{L}\gamma}{\mathrm{L}a}\right)$ among them when $\gamma + \delta i = \gamma$ and is positive, and this actual value was the only one termed artificial in old mathematics.

(C) The general logarithm always becomes the natural one, whenever the base $p + qi$ is taken $= e$ (so that $q = 0$ and $p = e$).

(D) If γ, δ, ϕ, q are so chosen that

$$\mathrm{L}r : \mathrm{L}\rho = \phi : (2n\pi + \psi),$$

for any particular value of n, then the general logarithm of an imaginary number, for an imaginary base, has always one actual value*.

(E) For these general logarithms the formulæ

$$(1) \qquad \overset{c}{\log}(ab) = \overset{c}{\log} a + \overset{c}{\log} b,$$

$$(2) \qquad \overset{c}{\log}\left(\frac{a}{b}\right) = \overset{c}{\log} a - \overset{c}{\log} b,$$

$$(3) \qquad \overset{c}{\log}(\sqrt[m]{a}) = \frac{\overset{c}{\log} a}{m},$$

still hold unconditionally, since they contain precisely the same number of values on the right as on the left, as follows immediately from the formulæ in (sect. 60) in conjunction with (No. I.).

On the contrary, the equation

$$\overset{c}{\log}(a^b) = b \,.\, \overset{c}{\log} a,$$

does *not* hold generally, as results also from (sect. 60). We must consequently either *never* apply this latter equation, or only with the greatest caution, as it only holds in a very restricted signification.

Section 63b.

If we now distinguish the *simplest value* $\frac{\mathrm{L}a}{\mathrm{L}c}$ of the general logarithm and denote it by $\mathrm{L}^c a$, we have unconditionally for these simplest values

$$(1) \qquad \mathrm{L}^c(ab) = \mathrm{L}^c a + \mathrm{L}^c b,$$

$$(2) \qquad \mathrm{L}^c\left(\frac{a}{b}\right) = \mathrm{L}^c a - \mathrm{L}^c b,$$

$$(3) \qquad \mathrm{L}^c(a^b) = b \,.\, \mathrm{L}^c a,$$

while

$$(4) \qquad \mathrm{L}^c(\sqrt[b]{a}) = \frac{\mathrm{L}^c a}{b},$$

only holds when $\sqrt[b]{a}$ represents the *simplest* of its values† (Comp. 57b.)

* As the simplest example of this case let $\gamma = p$, $\delta = q$, then $\rho = r$, $\psi = \phi$; and consequently the above condition is satisfied (for $n = 0$), and we have (from No. II.) $\overset{p+q.i}{\log}(p + q \,.\, i) = 1$, which is clearly correct. In the same manner if $\gamma = p^2 - q^2$ and $\delta = 2pq$, then $\rho = p^2 + q^2 = r^2$ and $\psi = 2\phi$, so that the formula (No. II.) now gives for $n = 0$,

$$\overset{p+q.i}{\log}(p^2 - q^2 + 2pq \,.\, i) = 2,$$

which is in this case perfectly correct, since we have really

$$(p + q \,.\, i)^2 = p^2 - q^2 + 2pq \,.\, i.$$

† If a, b, m are positive, these are at the same time the formulæ for actual logarithms, which were exhibited in (sect. 30).

Remark. We must however not omit to remark that the general logarithm assumes an inadmissible form, as we see from formulæ (No. II.), whenever, $Lr = 0$, and at the same time $\phi = 0$, that is, when $r^2 = p^2 + q^2 = 1$, and $\frac{p}{r} = 1$, $\frac{q}{r} = 0$, and therefore $p = 1$, $q = 0$, that is, when the base is supposed to be $= 1$.

And since, (by remark to sect. 58) Lr is inadmissible in calculation whenever $r = 0$, and therefore $p = q = 0$, it follows, *that logarithms are inadmissible in calculation,*

(*a*) when the logarithmands are 0;
(*b*) when the bases are 0 or 1.

Whenever, therefore, we meet $\overset{b}{\log} 0$, $\overset{0}{\log} b$ or $\overset{1}{\log} b$, in applications of general calculations, the calculation must immediately cease, and we must enter upon a particular investigation of what holds in these particular cases.

SECTION 64.

OF THE INFINITELY MULTIPLE-MEANING GENERAL POWER*.

We shall now consider the infinitely multiple-meaning general power a^x or $e^{x \log a}$, of which the *single-meaning* general power considered in (sect. 62) is the *simplest value*, and which must be understood by the name "general power" without any qualification being added. If we "calculate" it for $a = p + qi$ and $x = \alpha + \beta \cdot i$ we obtain

$$(\odot) \ldots (p+qi)^{\alpha+\beta i} = e^{\alpha \cdot Lr - (2n\pi+\phi)\beta} \times \left[K_{\beta \cdot Lr + \alpha(2n\pi+\phi)} + i \cdot S_{\beta \cdot Lr + \alpha(2n\pi+\phi)}\right],$$

where n represents successively zero and every positive and also every negative whole number†.

If we now consider the infinitely many values of this general power a^x, on the right in $(\odot)$, we find very easily that:

(1) They are all equal to one another, when x is positive or negative whole; and this single value then exactly coincides with the difference-power as defined in (sects. 24 and 25).

(2) These infinitely many values reduce to ν really different values, when $x = \frac{\mu}{\nu}$ and is positive or negative broken, but expressed in its lowest terms; and then these ν values exactly coincide with the ν values of $\sqrt[\nu]{(a^\mu)}$ in (sect. 57).

(3) Finally, the number of *different* values of a^x is really infinitely great, when x is actual and irrational, or when x is imaginary.

(4) Consequently, if the exponent x in a^x, that is, $(p+qi)^{\alpha+\beta i}$, is positive or negative, whole or zero, the simplest value of the

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* This is that power which is called the *general power* simply, in the Author's Instruction-Books.

† It is clear from what was said in the remark to (sect. 62) that we must here exclude the case in which the dignand becomes $= 0$.

general power does not differ from the infinitely multiple-meaning general power, i.e. the general power simply, and both accurately coincide with the difference-power of (sect. 24).

(5) But if the exponent x is positive or negative broken and $=\frac{\mu}{\nu}$, where $\frac{\mu}{\nu}$ is expressed in its lowest terms, then the simplest value of the general power differs from the infinitely multiple-meaning general power, i.e. the general power simply, in this respect, that the former *only* represent *one*, and that the *simplest* (sect. 57[b]) value, while the latter expresses *exactly all* the values of $\sqrt[\nu]{a^\mu}$.

Finally, since the general power (simply) is ambiguous, we must "calculate" with it with the same precaution, which must be used in calculating with all ambiguous expressions, which we have described in (sects. 37 and 41), and to which we again drew attention in (sect. 60).

If we consider e^x as the simplest value of the general power, it is precisely the natural power. But if we consider e^x as the general power (simply), it will have an infinite number of values, of which the natural is only one, and that the *simplest*.

We shall in future consider every power whose dignand is the number e as only having one meaning, so that it will always be the natural power of (sect. 49), unless the contrary be expressly mentioned.

If we now investigate the laws by which we may be able to "calculate" with these (infinitely multiple-meaning) general powers, in the sense of (sect. 6), we find the following results:

(1) $a^x \cdot a^z = a^{x+z} \cdot e^{2(mx+nz)\pi \cdot i}$;

(2) $a^x : a^z = a^{x-z} \cdot e^{2(mx+nz)\pi \cdot i}$;*

where m and n represent, independently of each other, zero, or any positive or negative whole number;

(3) $a^x \cdot b^x = (ab)^x$,

(4) $a^x : b^x = (a : b)^x$,

(5) $(a^x)^z = a^{xz} \cdot e^{2nz \cdot i}$,

where n represents zero and any positive or negative whole number.

These equations may be clearly and easily deduced from the preceding definition of the (infinitely multiple-meaning) general power, if we remark that we can obtain *all* the values of an (infinitely multiple-meaning) general power by multiplying one single determinate value of the same by *all* the values of 1^x, that is, by $e^{x \cdot \log 1}$, that is, by $e^{2nx\pi \cdot i}$.

---

* In the particular case that x and z are actual, we must substitute for the power $e^{2(mx+nz)\pi \cdot i}$ to the right in the formulæ (Nos. 1 and 2), the expression

$$K_{2(mx+nz)\pi} + i \cdot S_{2(mx+nz)\pi},$$

which is equal to it, in order to "calculate" the expression on the right.

Similarly for the expression on the right in (No. 5).

We may now draw a number of important conclusions:

(A) The equations (Nos. 3 and 4) which are commonly applied, contain precisely the same number of values to the right and left, and consequently their application leads to results that are necessarily correct, whatever the exponents may be, actual or imaginary.

(B) But if, instead of the complete formulæ (Nos. 1 and 2), we apply the usual formulæ which hold perfectly for *single-meaning* powers, viz. the formulæ

$$a^x . a^z = a^{x+z}, \text{ and } a^x : a^z = a^{x-z},$$

we run the chance of falling into great errors in *general calculations*. For if we substitute a^{x+z} for $a^x . a^z$, or a^{x-z} for $a^x : a^z$, we lose the greater part of the values of the given expressions, and only retain those which result from putting $m = n = 0$ in (Nos. 1 and 2).

(C) The case of the application of formula (No. 5) is perfectly analogous, if we namely employ in general calculations instead of this, the usual formula which holds for single-meaning, and therefore also for actual powers,

$$(a^x)^z = a^{xz},$$

and write a^{xz} simply for $(a^x)^z$.

(D) Conversely: if in general calculations, we substitute the product $a^x . a^z$ for the power a^{x+z}, or the quotient $a^x : a^z$ for the power a^{x-z}, or the power $(a^x)^z$ for the other power a^{xz}, then we have either substituted precisely the same values as before, or expressions which, although they possess many (infinitely many) more values than the expressions for which they were put, yet contain the values of the latter among their own, so that we at any rate lose no values by so doing.

Since all this is so extremely important, and since, if we neglect these truths, *it is no longer possible to calculate generally with powers* (or with general powers simply, and we are often quite unable to avoid so doing,) *and yet be sure of success*, we will here subjoin a few examples.

First Example. If we write

$$a^{\frac{2}{3}} . a^{\frac{5}{6}} = a^{\frac{2}{3}+\frac{5}{6}} = a^{\frac{3}{2}}.$$

then $a^{\frac{2}{3}}$ has three, and $a^{\frac{5}{6}}$ six values, and consequently the product $a^{\frac{2}{3}} . a^{\frac{5}{6}}$ has eighteen values, which are however three and three equal to one another, so that there remain only *six* really different values (i. e. unequal to one another). Now the expression on the right has only *two* different values, and the number would not be increased even if we were to write $a^{\frac{9}{6}}$ or $a^{\frac{27}{18}}$ for $a^{\frac{3}{2}}$. The above equation is therefore not correct in the sense of (sect. 3). But if we write according to (No. 1),

$$a^{\frac{2}{3}} . a^{\frac{5}{6}} = a^{\frac{3}{2}} . e^{2\left(\frac{2}{3}m + \frac{5}{6}n\right)\pi . i}$$

that is,

$$a^{\frac{2}{3}} \cdot a^{\frac{5}{6}} = a^{\frac{3}{2}} \cdot [\mathrm{K}_{(4m+5n)\pi:3} + i \cdot \mathrm{S}_{(4m+5n)\pi:3}];$$

or, if we diminish the argument (arc) by $2(m+n)\pi$, and write m for m, since m and n may be either positive or negative whole numbers,

$$a^{\frac{2}{3}} \cdot a^{\frac{5}{6}} = a^{\frac{3}{2}} \cdot [\mathrm{K}_{(2m-n)\pi:3} + i \cdot \mathrm{S}_{(2m-n)\pi:3}];$$

or, since $2m - n$ includes zero and *every* whole positive or negative number μ,

$$a^{\frac{2}{3}} \cdot a^{\frac{5}{6}} = a^{\frac{3}{2}} \cdot (\mathrm{K}_{\mu\pi:3} + i \cdot \mathrm{S}_{\mu\pi:3}),$$

then the expression on the right has also 6 values, and those precisely the same as the values on the left; and the equation is now perfectly correct.

Second Example. If we write

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{1} = a,$$

then we have *two* values on the left and only one single value on the right. The equation is therefore not correct in the sense of (sect. 3). But if we write (by No. 1),

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a \cdot e^{(m+n)\pi \cdot i} = a \cdot e^{\mu\pi \cdot i} = a \cdot (\mathrm{K}_{\mu\pi} + i \cdot \mathrm{S}_{\mu\pi}) = a \cdot \mathrm{K}_{\mu\pi}$$
$$= a\,(\pm 1) = \pm a,$$

we have then precisely the same two values on the right as on the left, and the equation is now perfectly correct.

Third Example. Sometimes we accidentally obtain as many values on the right as on the left; for example, if we write

$$a^{\frac{5}{6}} \cdot a^{\frac{2}{9}} = a^{\frac{5}{6}+\frac{2}{9}} = a^{\frac{19}{18}},$$

then we have 54 values on the left, which are however three and three equal to one another, so that they reduce to 18 really different values, while the expression on the left also represents all 18 values. In this case we have already accidentally obtained a perfectly correct equation without applying the improved formula (No. 1). But if we write by (No. 1),

$$a^{\frac{5}{6}} \cdot a^{\frac{2}{9}} = a^{\frac{19}{18}} \cdot e^{2\left(\frac{5}{6}m + \frac{2}{9}n\right)\pi \cdot i}$$
$$= a^{\frac{19}{18}} \cdot [\mathrm{K}_{2(15m+4n)\pi:18} + i \cdot \mathrm{S}_{2(15m+4n)\pi:18}],$$

then, since the second factor on the right is clearly always a value of $\sqrt[18]{1}$, we have no more nor less than these same 18 values on the right.

Fourth Example. This last is also the case in the equation

$$a^{\frac{5}{6}} \cdot a^{\frac{3}{4}} = a^{\frac{19}{12}},$$

since the 24 values of the product on the right are two and two equal to one another, and therefore reduce to 12.

Fifth Example. If we take

$$a^{\frac{1}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{1}{3}+\frac{2}{3}} = a^{1} = a,$$

we have nine values on the left, which reduce to 3, and only one on the right. The equation is therefore not correct in the sense of (sect. 3). But if we write by (No. 1)

$$a^{\frac{1}{3}} \cdot a^{\frac{2}{3}} = a \cdot e^{\frac{2}{3}(m+2n)\pi \cdot i} = a \cdot [\mathrm{K}_{2(m-n)\pi : 3} + i \cdot \mathrm{S}_{2(m-n)\pi : 3}],$$

(where the argument has been diminished by $2n\pi$, since the values of the series K and S are not altered by so doing,) then the expression on the right has also 3 values, being precisely the same as those of the product on the left.

Sixth Example. We will now take an example of the application of formula (No. 5). Let us write

$$(a^{\frac{2}{3}})^{\frac{3}{4}} = a^{\frac{2}{3} \cdot \frac{3}{4}} = a^{\frac{1}{2}}.$$

Now since $(a^{\frac{2}{3}})^{\frac{3}{4}}$ is perfectly equivalent to $\sqrt[4]{(a^{\frac{2}{3}})^{3}}$, and therefore to $\sqrt[4]{a^{2}}$, the expression on the left has 4 values, viz. $+\sqrt{a}$, $-\sqrt{a}$, $+i\sqrt{a}$, $-i\sqrt{a}$; while the expression $a^{\frac{1}{2}}$ on the right has only 2 of these values, and cannot therefore be unconditionally substituted for $(a^{\frac{2}{3}})^{\frac{3}{4}}$. But if we apply formula (No. 5), we obtain

$$(a^{\frac{2}{3}})^{\frac{4}{3}} = a^{\frac{1}{2}} \cdot e^{\frac{3}{2}n\pi \cdot i} = a^{\frac{1}{2}}(\mathrm{K}_{\frac{3}{2}n\pi} + i \cdot \mathrm{S}_{\frac{3}{2}n\pi});$$

and in this result the second factor on the right has for $n = 0, 1, 2, 3$, &c. &c. the four values $1, -i, -1, +i$, and if we multiply the 2 values of $a^{\frac{1}{2}}$, that is, $+\sqrt{a}$ and $-\sqrt{a}$, by these, we obtain 8 values, which reduce to 4, which are precisely the same 4 as are possessed by the expression on the left.

We thus see that the equation $(a^{x})^{z} = a^{xz}$ may *not* be *generally* applied but that the equation (No. 5) must take its place.

Section 65.

The following particular propositions may however be easily proved, viz.

If $\frac{m}{n}$ and $\frac{\mu}{\nu}$ are two fractions expressed in their lowest terms, and we take,

$$(1) \quad a^{\frac{m}{n}} \cdot a^{\frac{\mu}{\nu}} = a^{\frac{m}{n}+\frac{\mu}{\nu}} = a^{\frac{m\nu+n\mu}{n\nu}},$$

$$(2) \quad a^{\frac{m}{n}} : a^{\frac{\mu}{\nu}} = a^{\frac{m}{n}-\frac{\mu}{\nu}} = a^{\frac{m\nu-n\mu}{n\nu}},$$

then in both of these equations the expression on the left will always have the same $n\nu$ values and no more, as the expression on the right, provided n and ν are relative prime numbers, i.e. have no common divisor; and in this case the equations do not require the correction enunciated in (sect. 64, Nos. 1 and 2). But if n and ν have a common divisor τ, then the expression on the right has only $\frac{n\nu}{\tau}$ different values, and the formulæ (Nos. 1 and 2) of (sect. 64) must now again take the place of the above (Nos. 1 and 2), in order that the equations may be complete, and may hold universally*.

In the same way the equation

$$(3) \quad (a^{\frac{m}{n}})^{\frac{\mu}{\nu}} = a^{\frac{m}{n} \cdot \frac{\mu}{\nu}} = a^{\frac{m\mu}{n\nu}},$$

is also perfectly correct, and does not need the correction enunciated in (sect. 64, No. 5) in all the particular cases in which $\frac{m}{n}$ and $\frac{\mu}{\nu}$ are expressed in their lowest terms, and at the same time m and ν, as also n and μ have no common divisor.

Section 66.

OF THE MOST GENERAL LOGARITHMS.

The *most general* logarithm $\overset{a}{\log} b$ or $b ? a$ represents any expression x, for which the infinitely multiple-meaning general power a^x, that is $e^{x \log a}$, $= b$. Hence

$$\overset{a}{\log} b \ \text{ or } \ b ? a = \frac{\log b}{\log a},$$

$$\text{or} \qquad b ? a = \frac{Lb + 2n\pi i}{La + 2\nu\pi i},$$

where n and ν represent independently zero or any positive or negative number, and the most general logarithm has therefore an infinity times infinitely many values, among which are comprehended those of the general logarithm of (sect. 63).

Since the object of these pages is only to establish the ideas, and to prove the existence of firm interior connection and scientific unity in the whole of mathematical analysis, we shall not enter upon a farther investigation of these *most general* logarithms, and that especially because the investigation itself is neither very difficult, nor very profitable.

Section 67.

Let us finally endeavour to find a series

$$R = a + bx + cx^2 + dx^3 + \ldots$$

which proceeds according to whole powers of x, and which is equal to $\log(1+x)$ in the sense that $e^R = 1 + x$; we can accomplish the

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* Our idea of "equation" must be here remembered, according to which two forms (expressions) are equal to one another, when they may be unconditionally substituted for one another, with the consciousness that by so doing we cannot come into contradiction with the laws of operation in any way whatsoever.

solution of this problem by means of indeterminate coefficients in a more or less direct manner; that is, we can either put directly

$$e^a . e^{bx} . e^{cx^2} . e^{dx^3} \ldots = 1 + x,$$

that is,

$$e^a . \left(1 + bx + \frac{b^2 x^2}{2!} + \ldots\right)\left(1 + cx^2 + \frac{c^2 x^4}{2!} + \ldots\right)\left(1 + dx^3 + \frac{d^2 x^6}{2!} + \ldots\right) \ldots = 1 + x;$$

or we can endeavour to find all the series R_x which have in common with $\log(1 + x)$ the property that

$$R_x + R_z = R_{x+z+xz} \text{ or } 2R_x = R_{2x+x^2}.*$$

In the first case we find $e^a = 1$, that is, $a = \log 1$ or $a = 2n\pi . i$, and besides this $b = 1$, $c = -\frac{1}{2}$, $d = \frac{1}{3}$, $e = -\frac{1}{4}$, &c. &c.

In the other case we find $a = 0$, the coefficient b remains indeterminate, and we find the series

$$b\left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots\right)$$

which, agreeably to the condition made in this case, has the property of one of the values of every kind of logarithm, and not merely of the natural logarithm.

For the natural logarithm the *modulus* b will be equal to unity, so that {since, if $\log(1 + x)$ represents *one* of the values, then $\log(1 + x) + 2n\pi i$ represents *all* the infinitely many values of $\log(1 + x)$} we obtain in this way precisely the same result as we did in the first and more direct way.

We might also attempt the following method of deduction: making use of the binomial theorem, (which is already developed in (sect. 42) for a *positive* whole exponent z,) we have, as is shewn in (sect. 42),

$$(a) \quad (1 + x)^z = 1 + z . x + \frac{z(z-1)}{1 . 2} . x^2 + \frac{z(z-1)(z-2)}{1 . 2 . 3} . x^3 + \ldots$$

$$= 1 + \left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots\right) z + (\ldots) z^2 + \ldots$$

On the other hand, by (sect. 49)

$$(b) \quad (1 + x)^z = 1 + \{\log(1 + x)\} . z + \frac{\{\log(1 + x)\}^2}{2!} . z^2 + \ldots$$

By comparing these series with one another†, we obtain immediately

$$(\mathfrak{D}) \ldots \log(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots$$

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* For we have

$$\log(1 + x) + \log(1 + z) = \log\{(1 + x)(1 + z)\} = \log\{1 + (x + z + xz)\};$$

and, at least conditionally,

$$2\log(1 + x) = \log\{(1 + x)^2\} = \log\{1 + (2x + x^2)\}.$$

† This comparison is however not allowable, because both equations do not hold for any indeterminate z. There is therefore no need that the result ($\mathfrak{D}$) should be in any respect, and certainly not that it should be generally correct; and this is in fact the case; for in ($\mathfrak{D}$) we have infinitely many values on the left, but on the right one single infinite series, which has the property of the logarithm of $1 + x$. On the other hand, if we had here generally established and proved the general binomial theorem, as it is given below in (sect. 68), the series to the right in

and generally from the formula (b),

$$\log (1+x)_1 = \frac{(1+x)^z - 1}{z}, \text{ for } z = 0.$$

But we must not overlook with respect to this deduction, that

(1) z must be considered as representing a whole number in (sect. 42), and that therefore the present comparison does not necessarily give generally correct results, consequently either of the other two methods is for the moment to be preferred {for finding log $(1+x)$}*.

(2) That the infinite series on the right in (b) has only the property in common with log $(1+x)$, that if it be denoted by R, the power e^R gives *in infinitum*, with the exception of the two first terms $1+x$, all its terms $=0$.

(3) That this is also the case when we complete the series R, by the addition of its very first term $2n\pi.i$, so that we obtain

(I.) $\log(1+b) = 2n\pi.i + b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \frac{1}{4}b^4 + \ldots$

We must also observe with respect to the infinite series R which we have found for $\log(1+x)$, that, if in particular cases, where particular values in cyphers are given to x, the series R becomes divergent, it has no value, and is inadmissible in calculation; but if it is convergent, it has a value, which has the same property as $\log(1+x)$, and which will therefore always be one of the infinitely many values of $\log(1+x)$, in case this *single* value has not been completed so as to represent *all* the values by the addition of the very first term $2n\pi.i$.

Now whether this logarithmic series be in particular cases convergent or divergent, we can, as long as no value in cyphers is substituted for x, that is, as long as the series is still general, i.e. as long as x is a mere supporter of the operations, "calculate" with this series as securely as with $\log(1+x)$ itself, although the latter has a value for *any* actual or imaginary value of x, while the series has only a value when it is convergent. But as soon as one of the resulting series becomes,

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(*a*) would have had the additional factor 1^z, that is, $e^{2nz\pi.i}$, that is, $1+2n\pi i.z - 4n^2\pi^2.z^2+\ldots$; and if we completed the multiplication of the two series on the right which both proceed according to whole powers of z, we should find in the new series for $(1+x)^z$, when arranged according to z, the series

$$2n\pi.i + x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots$$

as the coefficient of z. Consequently if we now compared the series on the right in the results (*a*) and (*b*), we should have the generally correct equation

$$\log(1+x) = 2n\pi.i + x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots$$

which holds for every indeterminate x that may be considered as a supporter of the symbols of operation, and which also holds for every actual or imaginary x for which the series is convergent; and for every other actual or imaginary x is (not untrue but) inadmissible in calculation.

* This development has in other respects, when it has been performed with sufficient generality, (that is, not till *after* sect. 68), the great advantage of being the only one which gives, in a decided manner, the law according to which the terms of the logarithmic series proceed *in infinitum*.

for particular values in cyphers, numerical and divergent, it can be no longer employed for further applications. The remaining series which are deduced from that found for $\log(1+x)$, as for example,

$$\text{(II.)}\quad \log(1-x) = 2n\pi . i - x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \dots$$

and $$\text{(III.)}\quad \log\frac{(1+x)}{(1-x)} = 2n\pi . i + 2\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \dots\right)$$

and so on, present no difficulty, provided it be kept in view,

(*a*) That we do not calculate with magnitudes (quantities); but,

(*b*) That all expressions are merely symbolized operations, and must be regarded as supporters of certain properties, that, for example, $\log\frac{1+x}{1-x}$ represents the property that when e is potentiated by it the result is $\frac{1+x}{1-x}$. Now the series just found has the same property, that is, if we potentiate e by it, we have

$$e^{2n\pi i} . e^{2x} . e^{\frac{2}{3}x^3} . e^{\frac{2}{5}x^5} \dots$$

and the result is $$\frac{1+x}{1-x};$$

that is, $$(1+x) . \frac{1}{1-x};$$

that is, $$(1+x)(1+x+x^2+x^3+x^4+\dots)$$

that is, $$1+2x+2x^2+2x^3+2x^4+2x^5+2x^6+\dots \textit{ in infinitum.}$$

If the series then is convergent when numerical, and therefore has a value, then this value has the same property, i.e. it is then one of the values of $\log\frac{1+x}{1-x}$. But if the same series is divergent for another value of x, it is no longer adapted to exhibit one of the values of $\log\frac{1+x}{1-x}$; for it is altogether inadmissible in calculation.

The series for logarithms are employed for the purpose of calculating some logarithms of *actual* numbers; and they are made more quickly convergent for this purpose by, for example, proceeding as follows in order to calculate La, where a is conceived as positive; we put

$$La = m . L(\sqrt[m]{a}), \qquad \text{which by (No. I.)}$$
$$= m . \{(\sqrt[m]{a}-1) - \tfrac{1}{2}(\sqrt[m]{a}-1)^2 + \tfrac{1}{3}(\sqrt[m]{a}-1)^3 - \dots\}$$

while m can be taken as large as we please, in order that $\sqrt[m]{a}-1$ may be very small. But, *if we wished it*, we might employ these series for the direct calculation of the logarithm of any *imaginary* number. In order to explain this by an example, suppose we had to calculate $\log(P+Q.i)$ where P and Q are any actual, i.e. positive or negative, whole or broken numbers. If we now put $(P-1)+Q.i$ for x in the formula ($\mathfrak{D}$), and calculate r and ϕ from the equations,

$$r=+\sqrt{(P-1)^2+Q^2} \text{ and } K_\phi=\frac{P-1}{r}, \quad S_\phi=\frac{Q}{r};$$

so that $x=r.(K_\phi+i\,S_\phi)$, and therefore $x^n=r^n.(K_{n\phi}+i\,S_{n\phi})$; then we obtain from (No. I.)

$$\log(P+Qi)=\log 1+r.K_\phi-\tfrac{1}{2}r^2.K_{2\phi}+\tfrac{1}{3}r^3.K_{3\phi}-\tfrac{1}{4}r^4.K_{4\phi}+\ldots$$
$$+i.\{r\,S_\phi-\tfrac{1}{2}r^2.S_{2\phi}+\tfrac{1}{3}r^3.S_{3\phi}-\tfrac{1}{4}r^4.S_{4\phi}+\ldots\}$$

from which one of the values of $\log(P+Q.i)$ may be calculated when $r<1$, while, on account of the very first term $\log 1$ or $2n\pi.i$, we have at the same time *all* the values of $\log(P+Q.i)$ also. We might now propose a method for making the series in this last problem, more quickly convergent, e.g. by first arranging the calculation so that r may be as small as we choose*.

These indications may suffice. The reader will easily perceive that a great number of investigations might be here inserted, which would be interesting and would throw more light upon the views here adopted, but which we cannot enter upon without being prolix, and consequently frustrating our present object.

SECTION 68.

The question now arises whether the binomial theorem still holds for the infinitely multiple-meaning general powers. And we find that it is still true for these most general powers (i.e. for those powers which the author in his Instruction-Books has termed "general powers" simply) provided we give it the form †

$$(1+x)^z=1^z.\{1+z.x+\frac{z^{2|-1}}{2!}.x^2+\frac{z^{3|-1}}{3!}.x^3+\frac{z^{4|-1}}{4!}.x^4\ldots\}.$$

So that, by means of the factor 1^z, that is, $e^{z.\log 1}$, that is, $e^{2nz\pi.i}$, it represents precisely the same number of forms on the right, as $(1+x)^z$ does on the left; and this equation holds unconditionally, *as long as x is still general,* and therefore also when the series is convergent for particular values of x, while the same series (like all others) is quite useless when no longer general, but numerical and divergent.

In order to prove this we begin by denoting the infinite series,

$$(1)\quad 1+z.x+\frac{z^{2|-1}}{2!}.x^2+\frac{z^{3|-1}}{3!}.x^3+\ldots,$$

which is conceived as existing for any x, and any z, without re-

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* Since tables are already calculated in which the logarithms of all positive numbers are contained, the method given in (sect. 58) for finding $\log(P+Qi)$ is for any practical purpose to be preferred to the present one. We only wanted to shew that this way *could* be chosen.

† The expression $z^{n|d}$ represents a product of n factors, the first of which is z, and each successive one of which is formed from the one immediately preceding by the addition of d; that is, the product $z(z+d)(z+2d)\ldots\{z+(n-1)d\}$.

garding whether it is, or is not equal to the power $(1+z)^x$, by f_z, and then converting it by the process described in (sect. 42) into a series proceeding according to powers of z, which may be represented by

(2) $1 + X_1 . z + X_2 . z^2 + X_3 . z^3 + X_4 . z^4 + \ldots$

or R_z, so that we have

(3) $$f_z = R_z.$$

Then we prove by simple multiplication (as *Euler* did), that

(4) $$f_z . f_y = f_{z+y},$$

and conclude from this (by means of equation No. 3), that

(5) $$R_z . R_y = R_{z+y} \text{ also.}$$

Now from this last equation it follows, as has been already shewn (sect. 49), that

(6) $$X_2 = \frac{X_1^2}{2!},\ X_3 = \frac{X_1^3}{3!},\ X_4 = \frac{X_1^4}{4!}, \ldots *$$

That is, that X_2, X_3, X_4, &c. are infinite series which proceed according to powers of x, and which are obtained by potentiating the infinite series (proceeding according to powers of x) X_1, by 2, 3, 4, &c. respectively, and dividing the results by $2!$, $3!$, $4!$, &c., respectively. Now the series X_1 is found from the equation $f_z = R_z$, by subtracting the 1 on both sides, dividing by z, and finally putting $z = 0$. This gives

(7) $$X_1 = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots$$

that is, by (sect. 67),

(8) $$X_1 = \text{Lg}\,(1+x)$$

if by $\text{Lg}\,(1+x)$ we mean *one* of the forms of the natural logarithm of $1+x$.

By the equations (Nos. 6, and 7 or 8), the coefficients of the series No. 2, (or R_z) into which the series No. 1, (or f_z) allows of being transformed, may be found *in infinitum*, while we are at present perfectly ignorant of what other expression the series No. 1, (or f_z) is equal to.

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* By multiplying the two series R_z and R_y we obtain, as result, the series whose general term is

$$X_m . X_n . z^m . y^n.$$

Then we take the series R_{z+y}, that is, the series whose general term is $X_p(z+y)^p$, and develop the power $(z+y)^p$ for the different values of p, so that we find

$$X_{m+n} . \frac{(m+n)!}{m!\ n!} . z^m y^n,$$

for the general term of the series; and comparing the terms, on account of the equation (No. 5), we find

$$\frac{(m+n)!}{m!\ n!} X_{m+n} = X_m . X_n,$$

or for $n = 1$,

$$(m+1) X_{m+1} = X_m . X_1,$$

for any whole number m.

But now, on the other hand, by the definition of the general power,

$$\begin{aligned}(1+x)^z &= e^{z \cdot \log(1+x)} = e^{z \cdot [\log 1 + \mathrm{Lg}(1+x)]} \\ &= e^{z \cdot \log 1} \cdot e^{z \cdot \mathrm{Lg}(1+x)} = 1^z \cdot e^{z \cdot X_1} \\ &= 1^z \cdot \left(1 + X_1 z + \frac{X_1^2 z^2}{2!} + \frac{X_1^3 z^3}{3!} + \ldots\right) \\ &= 1^z \cdot R_z,\end{aligned}$$

that is, since $R_z = f_z$, (No. 3), perfectly generally,

$$(1+x)^z = 1^z \cdot f_z, \text{ that is}$$

$$(\odot) \ldots (1+x)^z = 1^z \cdot \{1 + z \cdot x + \frac{z^{2|-1}}{2!} x^2 + \frac{z^{3|-1}}{3!} x^3 + \ldots\}$$

which is the binomial theorem for the infinitely multiple-meaning general power, and where x and z are considered as perfectly arbitrary, and mere supporters of the symbols of operation, and may therefore be just as well actual as imaginary.

The whole proof, however, only shews that the infinite series represented by $(1+x)^z$, that is, by $e^{z \cdot \log(1+x)}$ may be transformed into the other infinite series, which we term the Binomial Series generally, agreeably to the laws of operation.

If we omitted the factor 1^z on the right hand in ($\odot$), and therefore selected the single form f_z from out of all the different ones, this would of course only express one of the values of $(1+x)^z$; we should then certainly have to investigate what value it is, and we could not therefore assert without further consideration, that we should thus obtain the *simplest* value of $(1+x)^z$, expressed by $e^{z \cdot L(1+x)}$.

But if x and z are actual, and the binomial series f_z at the same time convergent, there is then no doubt that the value of this binomial series f_z is always the *simplest* (viz. the actual) value of $(1+x)^z$.

We have here then exhibited the binomial theorem in its most general form, and proved it with perfect generality, conducting the proof with such care, that it must necessarily convey full conviction to every reader who in other respects admits the views here adopted; and we thus of course stand in contradiction with the celebrated *Cauchy* and all those, who only venture to apply the binomial theorem, i.e. this development of $(1+x)^z$ in the case that x is no longer general, but has already received an actual value in cyphers, and is small enough to make the binomial series a convergent (numerical) series.

SECTION 69.

The expression S_x (sect. 50) has for every actual or imaginary value of x of the form $p+qi$, always one, and never more than one value, which is also of the form $p+qi$. The same holds for the function K_x (sect. 50). The quotients

$$\frac{S_x}{K_x}, \quad \frac{K_x}{S_x}, \quad \frac{1}{K_x}, \text{ and } \frac{1}{S_x}$$

which are also functions of x, and have *always* one, and *never* more than one actual or imaginary value, may now be represented by the particular symbols,

$$\text{Tg}_x,\ \text{Cotg}_x,\ \text{Sec}_x, \text{ and } \text{Cosec}_x$$

respectively, and introduced into calculations.

Conversely; if we put $S_x = z$, that is (by sect. 51),

$$(1)\quad \frac{e^{xi} - e^{-xi}}{2i} = z,$$

then we find from this equation,

$$(2)\quad e^{xi} = z \cdot i \pm \sqrt{1 - z^2},$$

and therefore,

$$(3)\quad x = \frac{1}{i} \cdot \log(z \cdot i \pm \sqrt{1 - z^2}) = \frac{1}{i} \log(\pm\sqrt{1 - S_x^2} + i \cdot S_x),$$

that is, we find from this equation (No. 1) twice an infinite number of values of x (if we may so speak), since the logarithmand $z \cdot i \pm \sqrt{1 - z^2}$ has *two* values, and an infinite number of values of the natural logarithm exist for each logarithmand (sect. 58); and since all these values of x derived from (No. 3) actually satisfy the equation (No. 1), as we find by substitution, the equation (No. 3) gives *all* the arguments (arcs) x belonging to $S_x = z$, whether z, that is S_x, be actual or imaginary. But since we find *all* the values of $\log b$, by adding *all* the values of $\log 1$ or $2n\pi \cdot i$ to any one of the values of $\log b$, which may be denoted by $\text{Lg}\, b$, we have also

$$(4)\quad x = 2n\pi + \frac{1}{i} \cdot \text{Lg}\,(z \cdot i \pm \sqrt{1 - z^2}),$$

where n denotes zero and any positive or negative whole number, while Lg has *two* values, because the logarithmand is *double*-meaning.

We term any such value of x the *argument belonging to a given* S_x *or* z,* and denote it by $\frac{1}{S} z$, and therefore express the equation (No. 4) thus also:

$$\text{(I.)}\quad \frac{1}{S} z = \frac{1}{i} \cdot \log(z \cdot i \pm \sqrt{1 - z^2})$$
$$= 2n\pi + \frac{1}{i} \cdot \text{Lg}\,(z \cdot i \pm \sqrt{1 - z^2}).$$

If we now denote by $\frac{1}{K} z$, $\frac{1}{\text{Tg}} z$, $\frac{1}{\text{cotg}} z$, the infinitely many

* In Geometry, which has to be developed *after* Analysis, it is shewn that whenever S_x or K_x is actual and (absolutely) < 1, the *argument* x always expresses the arc of a circle whose radius is $= 1$, and whose abscissa and ordinate taken from the centre of the circle are S_x and K_x. For this reason it is generally usual to apply the name of a part to the whole, and to denote by the word *arc* that which we have just termed *argument*.

values of x for which $K_x = z$, or $Tg_x = z$, or $Cotg_x = z$, we then obtain in precisely the same manner:

$$\text{(II.)} \quad \frac{1}{K} z = \frac{1}{i} . \log (z \pm i . \sqrt{1-z^2}) = 2n\pi + \frac{1}{i} . \text{Lg} (z \pm i . \sqrt{1-z^2}).$$

$$\text{(III.)} \quad \frac{1}{Tg} z = \frac{1}{2i} . \log \frac{1+z.i}{1-z.i} = n\pi + \frac{1}{2i} \text{Lg} \frac{1+z.i}{1-z.i}.$$

$$\text{(IV.)} \quad \frac{1}{Cotg} z = \frac{1}{2i} . \log \frac{z+i}{z-i} = n\pi + \frac{1}{2i} \text{Lg} \frac{z+i}{z-i}, *$$

in all of which formulæ n represents zero or any positive or negative whole number, while Lg represents any one, but only one, of the values of the natural logarithm, and z has any (actual or imaginary) value of the form $p + q . i$.

The functions of z expressed by the symbols $\frac{1}{S} z$, $\frac{1}{K} z$, $\frac{1}{Tg} z$, and $\frac{1}{Cotg} z$ are therefore infinitely multiple-meaning *logarithmic* functions, while the functions of x represented by S_x, K_x, Tg_x, $Cotg_x$ are always single-meaning exponential functions of x, single-meaning, because they only include the (single-meaning) *natural* power.†

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* We have by (sect. 50):

$$(1) \quad e^{xi} = K_x + i . S_x;$$

$$(2) \quad e^{-xi} = K_x - i . S_x;$$

therefore by dividing (1) by (2),

$$(3) \quad e^{2xi} = \frac{K_x + i . S_x}{K_x - i . S_x},$$

and thence also,

$$(4) \quad e^{2xi} = \frac{1 + i . Tg_x}{1 - i . Tg_x} = \frac{Cotg_x + i}{Cotg_x - i};$$

whence follow the formulæ (Nos. I—IV.)

† We find afterwards that if $\frac{1}{S} z$ only represents one of these values,

$$\frac{1}{S} z = z + \frac{1}{2} . \frac{z^3}{3} + \frac{1.3}{2.4} . \frac{z^5}{5} + \frac{1.3.5}{2.4.6} . \frac{z^7}{7} + \dots$$

or, if we put $\frac{1}{S} z = x$, so that $S_x = z$,

$$(\odot) \dots x = z + \frac{1}{2} . \frac{z^3}{3} + \frac{1.3}{2.4} . \frac{z^5}{5} + \frac{1.3.5}{2.4.6} . \frac{z^7}{7} + \dots$$

where x represents only *one* of the values of the argument x belonging to the equation $S_x = z$. If we now solve this equation ($\odot$) with respect to z, so that z may be expressed in x, then since it has the form of an higher algebraical equation with respect to z of an infinite degree, it may therefore give infinitely many values of z. But if we concluded from this circumstance that S_x would therefore have infinitely many values for one given value of x, we should commit the same error, which was remarked upon in the Introduction (Ans. to Quest. 2, (c.) p. 7), i. e. we should convert a general proposition generally, which is against the laws of logic. We have namely found the equation ($\odot$) in such a manner as to be certain that whenever z represents the series S_x, the equation itself is correct (identical). But from this we can only conclude, that: Among all the values which when substituted for z, satisfy the equation ($\odot$), that one must be included which $= S_x$.

Now by (sect. 67, No. III.) we can transform the logarithm of $\frac{1+xi}{1-xi}$ into a series proceeding according to whole powers of x, and by this means the above formula (No. III.) is transformed into

(V.) $$\frac{1}{\mathrm{Tg}}z = n\pi + z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 - \tfrac{1}{7}z^7 + \ldots$$

We may employ this formula for the purpose of calculating the number π, which we defined above (sect. 53) as the least positive number for which

$$\mathrm{S}\tfrac{1}{2}\pi = 1 \quad \text{and} \quad \mathrm{K}\tfrac{1}{2}\pi = 0.$$

We calculate namely from these and from the formulæ (sect. 51, Nos. VIII. and IX.), viz.

$$\mathrm{S}\tfrac{1}{2}\nu = \sqrt{\frac{1-\mathrm{K}\nu}{2}}, \quad \text{and } \mathrm{K}\tfrac{1}{2}\nu = \sqrt{\frac{1+\mathrm{K}\nu}{2}},$$

by putting $\frac{1}{2}\pi$ for ν,

$$\mathrm{S}\tfrac{1}{4}\pi = \tfrac{1}{2}\sqrt{2}\,;\ \mathrm{K}\tfrac{1}{4}\pi = \tfrac{1}{2}\sqrt{2}, \text{ and therefore } \mathrm{Tg}\tfrac{1}{4}\pi = 1,$$

and hence from (No. V.),

$$\frac{1}{\mathrm{Tg}}1 = n\pi + 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \ldots$$

while one of these values of $\frac{1}{\mathrm{Tg}}1$ is the number $\frac{1}{4}\pi$.

Now since we already know from (sect. 53) that $\frac{1}{2}\pi$ lies between 1 and 2, and that therefore $\frac{1}{4}\pi$ must lie between $\frac{1}{2}$ and 1, we easily convince ourselves that we must take $n = 0$, in order to have that argument which is expressed by $\frac{1}{4}\pi$, since the value of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c. itself lies between $\frac{1}{2}$ and 1.

We pass over the means which may now be employed in order to express π by a more rapidly converging series {by putting a smaller value for z in (No. V.)}, since they present no theoretical difficulty whatever.

SECTION 70.

But since $\frac{1}{\mathrm{S}}x$, $\frac{1}{\mathrm{K}}x$, $\frac{1}{\mathrm{Tg}}x$, $\frac{1}{\mathrm{cotg}}x$ are infinitely multiple-meaning symbols, we must calculate with them with the same precaution as we had to observe in calculations with many-meaning or infinitely multiple-meaning powers, roots, and logarithms; i.e. we may *not*, e.g. put

$$(p \pm q)\cdot\frac{1}{\mathrm{S}}x \text{ for } p\cdot\frac{1}{\mathrm{S}}x \pm q\cdot\frac{1}{\mathrm{S}}x$$

generally, since such a substitution is only allowable when the factor $\frac{1}{S}x$ in each of the products on the right, is one and the same, that is, is a common factor, i. e. when $\frac{1}{S}x$ represents only one and the same of its values.

The formulæ which are generally employed in calculations with these functions, namely the formulæ:

$$(1)\quad \frac{1}{S}x + \frac{1}{S}z = \frac{1}{S}(x\sqrt{1-z^2} + z\sqrt{1-x^2});$$

$$(2)\quad \frac{1}{S}x - \frac{1}{S}z = \frac{1}{S}(x\sqrt{1-z^2} - z\sqrt{1-x^2});$$

$$(\text{III.})\quad \frac{1}{K}x + \frac{1}{K}z = \frac{1}{K}\{xz - \sqrt{(1-x^2)(1-z^2)}\};$$

$$(\text{IV.})\quad \frac{1}{K}x - \frac{1}{K}z = \frac{1}{K}\{xz + \sqrt{(1-x^2)(1-z^2)}\};$$

$$(\text{V.})\quad \frac{1}{Tg}x + \frac{1}{Tg}z = \frac{1}{Tg}\cdot\frac{x+z}{1-xz};$$

$$(\text{VI.})\quad \frac{1}{Tg}x - \frac{1}{Tg}z = \frac{1}{Tg}\cdot\frac{x-z}{1+xz};\quad \text{and so on,}$$

must all be further investigated, in order to see if they are really correct equations in the sense of (sect. 3) or whether they require correction.

Now upon an examination we find that:

Since all the combinations of the values of the roots $\sqrt{1-x^2}$, and $\sqrt{1-z^2}$ in the formulæ (Nos. 1 and 2) on the right must be taken, there will be more values on the right than on the left; and if we did not chuse to take all the combinations of the values of these roots, the expression on the right in (Nos. 1 and 2) would, on the one hand *not* have values which the expression on the left has, and on the other hand, would have values which do *not* occur on the left.

Thus if we denote the argument (arc) on the right in (No. 1) by β, we have

$$S_\beta = x\sqrt{1-z^2} + z\sqrt{1-x^2}, \text{ and therefore, either}$$
$$K_\beta = +\{-xz + \sqrt{(1-x^2)(1-z^2)}\},$$
$$\text{or } K_\beta = -\{-xz + \sqrt{(1-x^2)(1-z^2)}\}.$$

The expression on the right in (No. 1) contains then all the arguments (arcs) which belong to the first value of K_β and also all that belong to the second value of K_β, while the sum on the left only gives those values which correspond to the first value of K_β. Similarly with (No. 2). The two equations (Nos. 1 and 2) are therefore *not* correct equations in the sense of (sect. 3), and must consequently not be applied without the greatest caution.

On the contrary, the equations (Nos. III.—VI.) prove to be correct equations, have precisely the same number of values, and precisely the same values on each side, so that the definition of (sect. 3) (according to which, equal expressions are those which may be unconditionally substituted for each other) is fulfilled.

Let us now consider the following equations which frequently occur in the applications of analysis:

(3) $\frac{1}{S} x = \frac{1}{K} \sqrt{1-x^2}$;

(4) $\frac{1}{S} x = \frac{1}{Tg} \cdot \frac{x}{\sqrt{1-x^2}}$;

(5) $\frac{1}{K} x = \frac{1}{S} \cdot \sqrt{1-x^2}$;

(6) $\frac{1}{K} x = \frac{1}{Tg} \cdot \frac{\sqrt{1-x^2}}{x}$;

(7) $\frac{1}{Tg} x = \frac{1}{S} \cdot \frac{x}{\sqrt{1+x^2}}$;

(8) $\frac{1}{Tg} x = \frac{1}{K} \cdot \frac{1}{\sqrt{1+x^2}}$.

We find upon an examination into the correctness or incorrectness of these equations, that we must replace them by others, writing

(VII.) $\frac{1}{S} (\pm x) = \frac{1}{K} \sqrt{1-x^2}$, for (No. 3);

(VIII.) $\frac{1}{S} (\pm x) = \frac{1}{Tg} \cdot \frac{x}{\sqrt{1-x^2}}$, for (No. 4);

(IX.) $\frac{1}{K} (\pm x) = \frac{1}{S} \cdot \sqrt{1-x^2}$, for (No. 5);

(X.) $\frac{1}{K} (\pm x) = \frac{1}{Tg} \cdot \frac{\sqrt{1-x^2}}{x}$, for (No. 6);

(XI.) $\frac{1}{Tg} (\pm x) = \frac{1}{S} \cdot \frac{x}{\sqrt{1+x^2}}$, for (No. 7);

(XII.) $\frac{1}{Tg} (\pm x) = \frac{1}{K} \cdot \frac{1}{\sqrt{1+x^2}}$, for (No. 8);

in order to obtain correct equations which are applicable in general calculations and satisfy the definition in (sect. 3).

Equations (Nos. 1 and 2), finally, which we found were not correct, will be also transformed into correct equations which may be applied in general calculations with perfect security respecting the result, by writing them thus:

$$\text{I.}\quad \frac{1}{\mathrm{S}}(\pm x) + \frac{1}{\mathrm{S}}(\pm z) = \frac{1}{\mathrm{S}}\{x\sqrt{1-z^2} + z\sqrt{1-x^2}\}.$$

$$\text{II.}\quad \frac{1}{\mathrm{S}}(\pm x) + \frac{1}{\mathrm{S}}(\pm z) = \frac{1}{\mathrm{S}}\{x\sqrt{1-z^2} - z\sqrt{1-x^2}\}.$$

The equations (Nos. I.—XII.) here exhibited may be therefore applied in any calculations however general, with the certainty that they can never at any time lead to contradictions*.

In all these formulæ x and z are perfectly general and may be just as well actual as imaginary.

Section 71.

In conclusion we shall make the following remarks:

(1) There are functions of x (as $\sqrt{ax^2+bx+c}$ when a is negative and $4ac > b^2$) which have nothing but imaginary values for any actual value of x.

(2) There are functions of x, which always assume imaginary values for every actual value of x except one, as $p + \sqrt{-(x-a)^2}$ which is actual for the single value $x = a$, or two, as $p + \sqrt{-(x-a)^2(x-b)^2}$, which is actual for $x = a$ and also for $x = b$, or three, as $p + \sqrt{-(x-a)^2(x-b)^2(x-c)^2}$ for $x = a$, $x = b$, and $x = c$, and so on.

(3) We can also conceive functions of the form

$$f_x + \psi_x \cdot \sqrt{-(x-a)^2},$$

$$\text{or } f_x + \psi_x \cdot \sqrt{-(x-a)^2(x-b)^2}, \text{ and so on,}$$

which also only become actual once, or twice, or so on, for all actual values of x.

And in the same way we can conceive infinitely many exceedingly different forms, containing x, and therefore called functions of x, which are given implicitly or explicitly, in finite form or in the form of an infinite series, proceeding according to the powers of any general letter, and which enunciate the most different properties.

(4) All these functions of x can be transformed into series proceeding according to powers of z, where z represents a determinate function of x; and with these series, as with the functions themselves, we can always calculate unconditionally and securely, as long as x is conceived as perfectly general (as a mere supporter of the operations) *and we only apply such formulæ*

* The investigations here mentioned are easily made by substituting for the symbols $\frac{1}{\mathrm{S}}$, $\frac{1}{\mathrm{K}}$, and $\frac{1}{\mathrm{Tg}}$ their logarithmic forms (sect. 69), while care is taken only to apply those formulæ in calculating with logarithms which have been ascertained to be generally correct; e.g. *not* to apply the incorrect formula $\log(a^2) = 2\log a$, but the correct formula $\log(a^2) = 2\log(\pm a)$.

*in our calculations as hold generally;** that is, such as have been examined in the present essay, and either found correct, or completed.

(5) Next, *continuing our analysis*, we encounter expressions, called *definite integrals*, or expressions compounded of such definite integrals, which frequently represent what are termed *discontinuous functions*. From this point we must distinguish between (*a*) *continuous* and (*b*) *discontinuous* functions, and designate by the former phrase the expressions that we have been hitherto considering, i. e. *forms* which arise from *symbolized* addition, subtraction, multiplication, division, potentiation, radication, and logarithmation, repeated any number (or an infinite number) of times, provided only that the infinite series proceed *in infinitum* according to a determinate law, and also according to the power of a general letter of progression.

(6) For these *continuous* functions therefore holds the unconditional general calculation, that has been taught in the present Essay, in which we have not to regard the convergence of the general infinite series, nor the signification of the several letters, because this calculation has only to do (sect. 6) with general expressions, i. e. only with symbolized operations, i. e. only with *forms*.

(7) With reference to so called *discontinuous* functions, and generally with reference to expressions which do not appear till a more advanced stage of analysis, we must necessarily wait for their appearance; and then the analyst's duty is to exhibit as satisfactory a "theory" as possible of these new phenomena. And for this purpose the author has destined a *second Essay* of about the same size as the present.

We shall employ the space which we have left to shew how the doctrine of the *forms* hitherto developed, may be applied to the *comparison of magnitudes*.

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* Thus, if we wish to calculate correctly with powers *in general*, we may not apply the formulæ

$$a^x \cdot a^z = a^{x+z}, \text{ and } \frac{a^x}{a^z} = a^{x-z}, \text{ and } (a^x)^z = a^{xz},$$

because they do not hold generally; but we must in their stead employ the correct (i. e. the completed) formulæ given in (sect. 64). We notice this error once more in particular, because it is so much the custom to calculate generally with powers according to these same formulæ, which were found to hold for particular single-meaning powers. Otherwise, as we have seen, these formulæ of powers are not the only ones generally received in the art of calculating with letters, which do not hold generally.

APPENDIX.

OF MAGNITUDES.

Section 72.

(1) If magnitudes are to be compared by means of calculation, they must be expressed (or conceived) as *denominate numbers*, which are referred to one and the same denomination.

(2) Now there are originally, i. e. in reality, only denominate *whole* numbers, which can be reduced however by the multiplication or division of the indenominate numbers, to *lower* or *higher* units (denominations), e. g. *feet* to *inches*, or *inches* to *feet*.

(3) In this last operation of division we do not always arrive at whole numbers, and this shews e. g. that 5 shillings or 17 shillings cannot be expressed in pounds. But in order to be able to reduce denominate numbers in general, i. e. when they are not yet determined and given, to higher units (by general division of their indenominate numbers), it is not only allowable, but it is absolutely necessary, to introduce *broken* denominate numbers, so that we understand by the broken denominate number $\frac{a}{b}E$ the a^{th} multiple of the b^{th} part of the denomination (unit) E. The consequence of the introduction of the *broken* denominate number is, that we are enabled to transform lower units into higher ones (by division of their indenominate numbers), without having to take into consideration whether the quotient obtained by division is equal to a whole number, or is and remains a quotient *per se* (merely a symbolized division), so that this transformation of whole numbers can be performed in safety, *even when they are still entirely unknown;* while, conversely, every *whole* or *broken* denominate number may be reduced to a lower unit by multiplying its indenominate number,—as may be easily proved.

Section 73.

We now define:

(1) *Equal* magnitudes to be such as may be expressed by the same denominate number.

(2) The *greater* or the *less* magnitude to be that which is expressed by the *greater* or the *less* (positive, whole or broken) indenominate number, in the sense of (sect. 23), (always upon the hypothesis of a common denomination.)

From this it follows that very small and very great magnitudes are also expressed by very small and very great indenominate numbers (the latter conceived as in sect. 23), which are referred to a determinate denomination (unit).

SECTION 74.

Magnitudes allow and require—

(1) A conjoining of two or more so as to form a new and greater one;

(2) A removing of a part from a whole; that is, of one magnitude from a greater;

(3) A multiplication of one and the same magnitude, e. g. as when a magnitude is taken n fold; finally;

(4) A separation of one magnitude into a number, as n, of equal parts.

We express the result of the first operation as a denominate number, by *adding* (in the sense of sect. 3) the *in*denominate (whole or broken) numbers, and referring the sum to the same denomination.

The result of the second operation is obtained by *subtracting* (in the sense of sect. 3) the *in*denominate numbers.

The result of the third operation is obtained by *multiplying* (in the sense of sect. 11) the *in*denominate (whole or broken) number by n.

The result of the fourth operation is obtained by *dividing* (in the sense of sect. 11) the *in*denominate (whole or broken) number by n.

All upon the hypothesis that we have common denominations or units throughout.

In this manner all problems of magnitudes are reduced to operations with *in*denominate (whole or broken) numbers in the sense of the preceding chapters, and consequently all that has preceded respecting *numeric forms* may be here immediately and directly applied to the *comparison of magnitudes.*

Remark. This exhausts all the problems which can be proposed in the *general doctrine of magnitudes.* For in every problem in which we have, from the given manner in which things are connected, and the *magnitude* (quantitas) of given magnitudes, to draw conclusions respecting the *magnitude* (quantitas) of yet unknown magnitudes, the whole turns upon our expressing given magnitudes as given denominate numbers, and the unknown magnitudes as unknown denominate numbers (i. e. as denominate numbers, in which, although the denomination or unit has been already assumed and is therefore fully determined, the *indenominate* whole or broken numbers are still unknown, and are therefore for the time *denoted* by x, z, &c. &c., or more complex expressions in which the unknown numbers occur). Now after we have found two different *forms,* from the conditions of the problem, for one and the same magnitude, (i. e. for its *in*denominate number,) these latter must be equal to one another (in the sense of sect. 3), and thus we obtain the equations between the mere forms (*in*denominate whole or broken numbers), *which constitute the statement of the problem,* and which have afterwards to be solved with respect to the unknown expressions which occur in them, in order to obtain the values of the unknown expressions which satisfy these equations. If these also satisfy the remaining conditions of the problem (at the head of which stands the condition that these values be neither negative nor imaginary), we shall have found the required *indenominate* whole or broken numbers, and therefore also the unknown denominate numbers, i. e. the unknown magnitudes.*

Thus much concerning the *general doctrine of magnitudes.*

When we afterwards proceed to magnitudes of *space*, and treat of curves, and among others, the circle, and propose to ourselves the problem of expressing the circular arc x in terms of its ordinate y, for the case that the radius of the circle $=1$, we shall find between x and y the equation $y=S_x$, where S_x denotes the determinate infinite series in (sect. 50). In the same way it is found that the two series K_β and S_β treated in (sect. 50), represent, (whenever β is the length of a circular arc in the circle whose radius $=1$), certain straight lines placed in the circle, those namely which are pointed out and treated of under the names of *cosine* and *sine* in so-called elementary trigonometry.

The sine and cosine of so-called elementary trigonometry are consequently nothing but the values in cyphers of these general expressions in letters K_β and S_β, for those small and positive values of β, which express the length of an arc in the circle whose radius is $=1$. To rise from these values in cyphers to the expressions in letters, was a problem, which we certainly find solved in the history of mathematics, but which, like all those problems where the *form* of expressions is to be discovered and concluded on from mere values (in cyphers), can never at any time admit of a *perfectly satisfactory* solution.

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* We might also introduce *negative denominate* numbers; it would however soon be found that their field of operation would be very limited, and that their introduction is productive of much disadvantage, and of almost no advantage at all.

A LIST

OF THE

MATHEMATICAL WORKS

OF

PROFESSOR MARTIN OHM,

PHIL. D., KNIGHT OF THE RED EAGLE, &c.

N.B. Those works to which an asterisk is prefixed are in course of translation.

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1. *Kurzes, gründliches und leichtfassliches Rechenbuch.* A short, rigorous, and easy System of Cyphering, for the use of Schools. pp. xxxvi. 111. Maurer, Berlin. 1818.

2. *Versuch einer kurzen, gründichen und deutlichen, auch Nichtmathematikern verständlichen Anweisung 10—14jährige Knaben zu einem leichten, gründichen und wissenschaftlichen Studium der Mathematik fähig zu machen.* An attempt at a short, rigorous, and clear method, intelligible to non-mathematicians, of making school-boys of from 10 to 14 years of age capable of a rigorous and scientific study of mathematics; intended as an Introduction to his works on Elementary Mathematics. pp. xxiv. 160. Riemann, Berlin. 1827.

3. *Elementar-Zahlenlehre.* Elements of the Doctrine of Numbers (Algebra) for the use of Schools and Colleges and for self-instruction, with an appendix containing a sketch of the foundations of a General Doctrine of Magnitudes. pp. xxii. 234. Palm and Encke, Erlangen. 1816.

* 4. *Die Reine Elementar-Mathematik.* Elements of Pure Mathematics, intended for middle and upper Schools, and for self-instruction, with a very great number of examples. 3 vols. *Second Edition.* Jonas, Berlin. 1834.

* Vol. I. Also under the title: Arithmetic as far as the higher equations. pp. xvi. 476.

* Vol. II. Also under the title: Plane Geometry, together with the continuation of Arithmetic, and Plane and Analytical Trigonometry. pp. xii. 436.

* Vol. III. Also under the title: Solid Geometry, including Spherical Trigonometry, Descriptive Geometry, Projection of Shadows, and Perspective. pp. xii. 340.

5. *Die analytische und höhere Geometrie in ihren Elementen.* Elements of Analytical and higher Geometry, with especial regard to the Theory of Conic Sections; for his Professorial Lectures, the use of Universities and Self-instruction.

Also under the title: Exercises in Algebra, Geometry and Trigonometry, under the name of Analytical Geometry; a continuation of the Elements of Pure Mathematics. pp. xi. 370. Riemann, Berlin. 1826.

6. *Versuch eines vollkommen consequenten Systems der Mathematik.* Attempt at a perfectly consequential System of Mathematics.

Vols. I. II. Also under the title: Instruction-Book in Lower Analysis. *Second Edition.* Riemann, Berlin.

Vols. III.—VII. Also under the title: Instruction-Book in Higher Analysis. Riemann, Berlin.

* Vol. I. Arithmetic and Algebra. pp. xxxiv. 418. Anno 1828.

Vol. II. Algebra and Analysis Finitorum. pp. xxx. 455. Anno 1829.

Vol. III. Differential Calculus. pp. xxx. 285. Anno 1829.

Vol. IV. Differential and Integral Calculus, with 54 tables of Integrals. pp. x. 234. Appendix pp. 99. Tables pp. 54. Anno 1830.

Vol. V. (pp. xx. 376. Anno 1831). Vol. VI. (pp. xii. 367, Anno 1832). Vol. VII. (pp. x. 208. Appendix pp. 159. Anno 1833). Continuation of the Differential and Integral Calculus, Differential Equations, the most General Doctrine of Maxima and Minima (Calculus of Variations), with a great number of examples.

*** Five more volumes are required to complete this work.

7. *Die Lehre vom Grössten und Kleinsten.* The Doctrine of Maxima and Minima, with an Introduction and an Appendix, the former containing the necessary propositions of the Differential and Integral Calculus, and the latter a somewhat more general Calculus of Variations. pp. xviii. 330. Riemann, Berlin. 1825.

* 8. *Lehrbuch der Mechanik.* Instruction-Book in Mechanics, together with the necessary Doctrines of Higher Analysis and Higher Geometry, with very numerous examples. Enslin, Berlin.

* Vol. I. Mechanics of an Atom, with the General Introduction. pp. xvi. 475. Anno 1836.

* Vol. II. Statics of a Rigid System. pp. xiv. 490. Anno 1837.

* Vol. III. Dynamics of a Rigid System. pp. xvi. 542. Anno 1838.

* 9. *Der Geist der mathematischen Analysis, und ihr Verhältniss zur Schule.* The Spirit of Mathematical Analysis and its relation to a Logical System, intended also as Appendix and Commentary to his Instruction-Books. pp. xvi. 189. Duncker and Humblot, Berlin. 1842. (*The present work.*)

10. *Aufsätze aus dem Gebiete der höheren Mathematik.* Tracts upon some parts of Higher Mathematics. pp. ix. 103. Reimer, Berlin.

11. *Zwei Abhandlungen.* Two Essays: (*a*) Communication of a new analytical discovery. (*b*) Means for forming a more correct judgment of the works of Prof. Dirksen of Berlin. pp. 24. Jonas, Berlin. 1831.

* 12. *Lehrbuch für den gesammten Mathematischen Elementar-Unterricht.* Instruction-Book in the whole of Elementary Mathematics, for middle and higher Schools. *Third Edition.* pp. viii. 232. F. Volckmar, Leipzig. 1842.

* 13. *Lehrbuch der gesammten höheren Mathematik.* Instruction-Book in the whole of higher Mathematics, for the use of the upper classes in superior Schools, and of the Universities, also for self-instruction, with numerous examples. Volckmar, Leipzig. 1839.

* Vol. I. Analysis Finitorum or Higher Algebra, Elements of Higher Geometry, the Differential Calculus, and its applications. pp. xvi. 476.

* Vol. II. Integral Calculus, Calculus of Variations, Calculus of Finite Sums and Differences, and its applications to Geometry and Analysis. pp. xii. 489.

* 14. *Kurzes Elementar-Lehrbuch der gesammten mechanischen Wissenschaften.* A Short Elementary Instruction-Book for all the Mechanical Sciences, for middle, higher, and commercial Schools, with many examples. pp. xii. 252. Enslin, Berlin. 1840.

*** It is intended that the next published Translations of these works shall consist of "Arithmetic and Algebra", as contained in the first and part of the second volume of the "System" (No. 6), followed by the "Instruction-Book in Higher Mathematics" (No. 13), to which it will serve as an introduction. The Manuscript of the "Arithmetic and Algebra" is already complete.

THE END.